# Asymptotic behaviour of the spectrum of a waveguide with distant perturbations

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We consider the waveguide modelled by a $n$-dimensional infinite tube. The operator we study is the Dirichlet Laplacian perturbed by two distant perturbations. The perturbations are described by arbitrary abstract operators "localized" in a certain sense, and the distance between their "supports" tends to infinity. We study the asymptotic behaviour of the discrete spectrum of such system. The main results are a convergence theorem and the asymptotics expansions for the eigenvalues. The asymptotic behaviour of the associated eigenfunctions is described as well. We also provide some particular examples of the distant perturbations. The examples are the potential, second order differential operator, magnetic Schrödinger operator, curved and deformed waveguide, delta interaction, and integral operator.

## Introduction

A multiple well problem for the Schrödinger operator attracted much attention of many researches. A lot of works were devoted to the studying of the semi-classical case (see, for instance, [13], [8], [19], and references therein). Quite similar problems deal with the case when the small parameter at higher derivative is replaced by a large distance between wells. We mention papers [11], [13], [15] as well as the book [9, Sec. 8.6] devoted to such problems (see also bibliography of these works). The main result of these works was a description of the asymptotic behaviour of the eigenvalues and the eigenfunctions as the distance between wells tended to infinity. In [12] a double-well problem for the Dirac operator with large distance between wells was studied. The convergence and certain asymptotic results were established. We also mention the paper [17], where the usual potential was replaced by a delta-potential supported by a curve. The results of this paper imply the asymptotic estimate for the lowest spectral gap in the case the curve

consists of several disjoint components and the distances between components tend to infinity.

One of the ways of further developing of the mentioned problems is to consider them not in $\mathbb{R}^n$, but in some other unbounded domains. A good candidate is an infinite tube, since such domain arises in the waveguide theory. One of such problems has already been treated in [3], where we considered a quantum waveguide with two distant windows. The waveguide was a two-dimensional infinite strip, where the Dirichlet Laplacian was considered. The perturbation was two segments of the same finite length on the boundary with Neumann boundary condition. As the distance between the windows tended to infinity, a convergence result and the asymptotic expansions for the eigenvalues and the eigenfunctions were obtained. The technique used in [3] employed essentially the symmetry of the problem.

In the present paper we study a situation which is more general in comparison with the articles cited. Namely, we deal with a $n$-dimensional infinite tube where the Dirichlet Laplacian $-\Delta^{(D)}$ is considered. The perturbation is two arbitrary operators $\mathcal{L}_\pm$ "localized" in a certain sense. The distance between their "supports" is assumed to be a large parameter. In what follows we will call such perturbations as distant perturbations. The precise description will be given in the next section; we only say here that a lot of interesting examples are particular cases of these operators (see Sec. 7).

Our main results are as follows. First we prove the convergence result for the eigenvalues of the perturbed operator, and show that the limiting values for these eigenvalues are the discrete eigenvalues of the limiting operators $-\Delta^{(D)} + \mathcal{L}_\pm$ and the threshold of the essential spectrum. The most nontrivial result of the article is the asymptotic expansions for the perturbed eigenvalues. Namely, we obtain a scalar equation for these eigenvalues. Basing on this equation, we construct the leading terms of the asymptotic expansions for the perturbed eigenvalues. We also characterize the asymptotic behaviour of the perturbed eigenfunctions.

If the distant perturbations are two same wells, it is well-known that each limiting eigenvalue splits into pair of two perturbed eigenvalues one of which being larger than a limiting eigenvalue while the other being less. It is also known that the leading terms of the asymptotics of these eigenvalues are exponentially small w.r.t. the distance between well, and have the same modulus but different signs. In the present article we show that the similar phenomenon occur in the general situation as well (see Theorem 1.8).

Let us also discuss the technique used in the paper. The kernel of the approach is a scheme which allows us to reduce the eigenvalue equation for the perturbed operator to an equivalent operator equation in a special Hilbert space. The main advantage of such reduction is that in the final equation the original distant perturbations are replaced by an operator which is meromorphic w.r.t. the spectral parameter and is multiplied by a small parameter. We solve this equation explicitly by the modification of the Birman-Schwinger technique suggested in [10], and in this way we obtain the described results. We stress that our approach requires

no symmetry restriction in contrast to [3]. We should also say that the kind of boundary condition is inessential, and for instance similar problem for the Neumann Laplacian can be solved effectively, too. Moreover, our approach can be applied to other problems with distant perturbations not covered by the problem considered here.

The article is organized as follows. In the next section we give precise statement of the problem and formulate the main results. In the second section we prove that the essential spectrum of the perturbed operators is invariant w.r.t. the perturbations and occupies a real semi-axis, while the discrete spectrum contains finitely many eigenvalues. The third section is devoted to the studying of the limiting operators; we collect there some preliminary facts required for the proof of the main results. In the fourth section we provide the aforementioned scheme transforming the original perturbed eigenvalue equation to an equivalent operator equation. Then we solve this equation explicitly. The fifth section is devoted to the proof of the convergence result. The asymptotic formulas for the perturbed eigenelements are established in the sixth section. The final seventh section contains some examples of the operators $\mathcal{L}_\pm$.

# 1 Statement of the problem and formulation of the results

Let $x = (x_1, x')$ and $x' = (x_2, \ldots, x_n)$ be Cartesian coordinates in $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$, respectively, and let $\omega$ be a bounded domain in $\mathbb{R}^{n-1}$ having infinitely differentiable boundary. We assume that $n \geqslant 2$. By $\Pi$ we denote an infinite tube $\mathbb{R} \times \omega$. Given any bounded domain $Q \subset \Pi$, by $L_2(\Pi, Q)$ we denote the subset of the functions from $L_2(\Pi)$ whose support lies inside $\overline{Q}$. For any domain $\Omega \subseteq \Pi$ and $(n-1)$-dimensional manifold $S \subset \overline{\Omega}$ the symbol $W_{2,0}^j(\Omega, S)$ will indicate the subset of the functions from $W_2^j(\Omega)$ vanishing on $S$. If $S = \partial\Omega$, we will write shortly $W_{2,0}^j(\Omega)$.

Let $\Omega_\pm$ be a pair of bounded subdomains of $\Pi$ defined as $\Omega_\pm := (-a_\pm, a_\pm) \times \omega$, where $a_\pm \in \mathbb{R}$ are fixed positive numbers. We let $\gamma_\pm := (-a_\pm, a_\pm) \times \partial\omega$. By $\mathcal{L}_\pm$ we denote a pair of bounded linear operators from $W_{2,0}^j(\Omega_\pm, \gamma_\pm)$ into $L_2(\Pi, \Omega_\pm)$. We assume that for all $u_1, u_2 \in W_{2,0}^j(\Omega_\pm, \gamma_\pm)$ the identity

$$(\mathcal{L}_\pm u_1, u_2)_{L_2(\Omega_\pm)} = (u_1, \mathcal{L}_\pm u_2)_{L_2(\Omega_\pm)} \tag{1.1}$$

holds true. We also suppose that the operators $\mathcal{L}_\pm$ satisfy the estimate

$$(\mathcal{L}_\pm u, u)_{L_2(\Omega_\pm)} \geqslant -c_0 \|\nabla u\|^2_{L_2(\Omega_\pm)} - c_1 \|u\|^2_{L_2(\Omega_\pm)} \tag{1.2}$$

for all $u \in W_{2,0}^2(\Omega_\pm, \gamma_\pm)$, where the constants $c_0$, $c_1$ are independent of $u$, and

$$c_0 < 1. \tag{1.3}$$

Since the restriction of each function $u \in W_{2,0}^2(\Pi)$ on $\Omega_\pm$ belongs to $W_{2,0}^2(\Omega_\pm, \gamma_\pm)$, we can also regard the operators $\mathcal{L}_\pm$ as unbounded ones in $L_2(\Pi)$ with the domain $W_{2,0}^2(\Pi)$.

By $\mathcal{S}(a)$ we denote a shift operator in $L_2(\Pi)$ acting as $(\mathcal{S}(a)u)(x) := u(x_1 + a, x')$, and for any $l > 0$ we introduce the operator

$$\mathcal{L}_l := \mathcal{S}(l)\mathcal{L}_-\mathcal{S}(-l) + \mathcal{S}(-l)\mathcal{L}_+\mathcal{S}(l).$$

Clearly, the operator $\mathcal{L}_l$ depends on values its argument takes on the set $\{x : (x_1 + l, x') \in \Omega_-\} \cup \{x : (x_1 - l, x') \in \Omega_+\}$. As $l \to +\infty$, this set consists of two components separated by the distance $2l$. This is why we can regard the operator $\mathcal{L}_l$ as the distant perturbations formed by $\mathcal{L}_-$ and $\mathcal{L}_+$.

The main object of our study is the spectrum of the operator $\mathcal{H}_l := -\Delta^{(D)} + \mathcal{L}_l$ in $L_2(\Pi)$ with domain $W_{2,0}^2(\Pi)$. Here $-\Delta^{(D)}$ indicates the Laplacian in $L_2(\Pi)$ with the domain $W_{2,0}^2(\Pi)$.

We suppose that the operators $\mathcal{L}_\pm$ are so that the operator $\mathcal{H}_l$ is self-adjoint. The main aim of this paper is to study the behaviour of the spectrum of $\mathcal{H}_l$ as $l \to +\infty$.

In order to formulate the main results we need to introduce additional notations. We will employ the symbols $\sigma(\cdot)$, $\sigma_{\mathrm{ess}}(\cdot)$, $\sigma_{\mathrm{disc}}(\cdot)$ to indicate the spectrum, the essential and discrete one of an operator. We denote $\mathcal{H}_\pm := -\Delta^{(D)} + \mathcal{L}_\pm$, and suppose that these operators with domain $W_{2,0}^2(\Pi)$ are self-adjoint in $L_2(\Pi)$.

*Remark* 1.1. We note that the assumptions (1.1), (1.2), (1.3) do not imply the self-adjointness of $\mathcal{H}_l$ and $\mathcal{H}_\pm$, and we can employ here neither KLMN theorem no Kato-Rellich theorem. At the same time, if the operators $\mathcal{L}_\pm$ satisfy stricter assumption and are $-\Delta^{(D)}$-bounded with the bound less than one, it implies the self-adjointness of $\mathcal{H}_l$ and $\mathcal{H}_\pm$.

Let $\nu_1 > 0$ be the ground state of the Dirichlet Laplacian in $\omega$.

Our first result is

**Theorem 1.1.** *The essential spectra of the operators $\mathcal{H}_l$, $\mathcal{H}_+$, $\mathcal{H}_-$ coincide with the semi-axis $[\nu_1, +\infty)$. The discrete spectra of the operator $\mathcal{H}_l$, $\mathcal{H}_+$, $\mathcal{H}_-$ consist of finitely many real eigenvalues.*

We denote $\sigma_* := \sigma_{\mathrm{disc}}(\mathcal{H}_-) \cup \sigma_{\mathrm{disc}}(\mathcal{H}_+)$. Let $\lambda_* \in \sigma_*$ be a $p_-$-multiple eigenvalue of $\mathcal{H}_-$ and $p_+$-multiple eigenvalue of $\mathcal{H}_+$, where $p_\pm$ is taken being zero if $\lambda_* \notin \sigma_{\mathrm{disc}}(\mathcal{H}_\pm)$. In this case we will say that $\lambda_*$ is $(p_- + p_+)$-multiple.

**Theorem 1.2.** *Each discrete eigenvalue of $\mathcal{H}_l$ converges to one of the numbers from $\sigma_*$ or to $\nu_1$ as $l \to +\infty$.*

**Theorem 1.3.** *If $\lambda_* \in \sigma_*$ is $(p_- + p_+)$-multiple, the total multiplicity of the eigenvalues of $\mathcal{H}_l$ converging to $\lambda_*$ equals $p_- + p_+$.*

In what follows we will employ symbols $(\cdot, \cdot)_X$ and $\|\cdot\|_X$ to indicate the inner product and the norm in a Hilbert space $X$.

Suppose that $\lambda_* \in \sigma_*$ is $(p_- + p_+)$-multiple, and $\psi_i^\pm$, $i = 1, \ldots, p_\pm$, are the eigenfunctions of $\mathcal{H}_\pm$ associated with $\lambda_*$ and orthonormalized in $L_2(\Pi)$. If $p_- \geqslant 1$, we denote

$$\boldsymbol{\phi}_i(\cdot, l) := (0, \mathcal{L}_+\mathcal{S}(2l)\psi_i^-) \in L_2(\Omega_-) \oplus L_2(\Omega_+),$$

$$\mathcal{T}_1^{(i)}\boldsymbol{f} := (f_-,\psi_i^-)_{L_2(\Omega_-)},\quad i=1,\ldots,p_-,$$

where $\boldsymbol{f} := (f_-,f_+)\in L_2(\Omega_-)\oplus L_2(\Omega_+)$. If $p_+\geqslant 1$, we denote

$$\begin{aligned}\boldsymbol{\phi}_{i+p_-}(\cdot,l) &:= (\mathcal{L}_-\mathcal{S}(-2l)\psi_i^+,0)\in L_2(\Omega_-)\oplus L_2(\Omega_+),\\ \mathcal{T}_1^{(i+p_-)}\boldsymbol{f} &:= (f_+,\psi_i^+)_{L_2(\Omega_+)},\quad i=1,\ldots,p_+.\end{aligned}$$

In the fourth section we will show that the operator

$$\mathcal{T}_2(\lambda,l)\boldsymbol{f} := \big(\mathcal{L}_-\mathcal{S}(-2l)(\mathcal{H}_+-\lambda)^{-1}f_+,\mathcal{L}_+\mathcal{S}(2l)(\mathcal{H}_--\lambda)^{-1}f_-\big) \tag{1.4}$$

in $L_2(\Omega_-)\oplus L_2(\Omega_+)$ satisfies the relation

$$\mathcal{T}_2(\lambda,l) = -\frac{1}{\lambda-\lambda_*}\sum_{i=1}^{p}\boldsymbol{\phi}_i(\cdot,l)\mathcal{T}_1^{(i)}+\mathcal{T}_3(\lambda,l), \tag{1.5}$$

for $\lambda$ close to $\lambda_*$, where $p:=p_-+p_+$, and the norm of $\mathcal{T}_3$ tends to zero as $l\to+\infty$ uniformly in $l$. We introduce the matrix

$$\mathrm{A}(\lambda,l) := \begin{pmatrix} A_{11}(\lambda,l) & \ldots & A_{1p}(\lambda,l)\\ \vdots & & \vdots\\ A_{p1}(\lambda,l) & \ldots & A_{pp}(\lambda,l)\end{pmatrix},$$

where $A_{ij}(\lambda,l) := \mathcal{T}_1^{(i)}(\mathrm{I}+\mathcal{T}_3(\lambda,l))^{-1}\boldsymbol{\phi}_j(\cdot,l)$.

**Theorem 1.4.** *Let $\lambda_*\in\sigma_*$ be $(p_-+p_+)$-multiple, and let $\lambda_i=\lambda_i(l)\xrightarrow[l\to+\infty]{}\lambda_*$, $i=1,\ldots,p$, $p:=p_-+p_+$, be the eigenvalues of $\mathcal{H}_l$ taken counting multiplicity and ordered as follows*

$$0\leqslant|\lambda_1(l)-\lambda_*|\leqslant|\lambda_2(l)-\lambda_*|\leqslant\ldots\leqslant|\lambda_p(l)-\lambda_*|. \tag{1.6}$$

*These eigenvalues solve the equation*

$$\det\big((\lambda-\lambda_*)\mathrm{E}-\mathrm{A}(\lambda,l)\big)=0, \tag{1.7}$$

*and satisfy the asymptotic formulas*

$$\lambda_i(l)=\lambda_*+\tau_i(l)\left(1+\mathcal{O}\left(l^{\frac{2}{p}}\mathrm{e}^{-\frac{4l}{p}\sqrt{\nu_1-\lambda_*}}\right)\right),\quad l\to+\infty. \tag{1.8}$$

*Here*

$$\tau_i=\tau_i(l)=\mathcal{O}(\mathrm{e}^{-2l\sqrt{\nu_1-\lambda_*}}),\quad l\to+\infty, \tag{1.9}$$

*are the zeroes of the polynomial* $\det\big(\tau\mathrm{E}-\mathrm{A}(\lambda_*,l)\big)$ *taken counting multiplicity and ordered as follows*

$$0\leqslant|\tau_1(l)|\leqslant|\tau_2(l)|\leqslant\ldots\leqslant|\tau_p(l)|. \tag{1.10}$$

*The eigenfunctions associated with $\lambda_i$ satisfy the asymptotic representation*

$$\psi_i(x,l) = \sum_{i=1}^{p_-} k_{i,j}\psi_j^-(x_1+l,x') + \sum_{i=1}^{p_+} k_{i,j+p_-}\psi_j^+(x_1-l,x') + \mathcal{O}(\mathrm{e}^{-2l\sqrt{\nu_1-\lambda_*}}), \quad l\to+\infty, \tag{1.11}$$

*in $W_2^2(\Pi)$-norm. The numbers $k_{i,j}$ are the components of the vectors*

$$\boldsymbol{k}_i = \boldsymbol{k}_i(l) = (k_{i,1}(l)\dots k_{i,p}(l))^t$$

*solving the system*

$$\big((\lambda-\lambda_*)\mathrm{E} - \mathrm{A}(\lambda,l)\big)\boldsymbol{k} = 0, \tag{1.12}$$

*for $\lambda = \lambda_i(l)$, and satisfying the condition*

$$(\boldsymbol{k}_i,\boldsymbol{k}_j)_{\mathbb{C}^p} = \begin{cases} 1, & \text{if} \quad i=j, \\ \mathcal{O}(l\mathrm{e}^{-2l\sqrt{\nu_1-\lambda_*}}), & \text{if} \quad i\neq j. \end{cases} \tag{1.13}$$

According to this theorem, the leading terms of the asymptotics expansions for the eigenvalues $\lambda_i$ are determined by the matrix $\mathrm{A}(\lambda_*,l)$. At the same time, in applications it could be quiet complicated to calculate this matrix explicitly. This is why in the following theorems we provide one more way of calculating the asymptotics expansions.

We will say that a square matrix $\mathrm{A}(l)$ satisfies the condition (A), if it is diagonalizable and the determinant of the matrix formed by the normalized eigenvectors of $\mathrm{A}(l)$ is separated from zero uniformly in $l$ large enough.

**Theorem 1.5.** *Let the hypothesis of Theorem 1.4 hold true. Suppose that the matrix $\mathrm{A}(\lambda_*,l)$ can be represented as*

$$\mathrm{A}(\lambda_*,l) = \mathrm{A}_0(l) + \mathrm{A}_1(l), \tag{1.14}$$

*where the matrix $\mathrm{A}_0$ satisfies the condition (A), and $\|\mathrm{A}_1(l)\|\to 0$ as $l\to+\infty$. In this case the eigenvalues $\lambda_i$ of $\mathcal{H}_l$ satisfy the asymptotic formulas*

$$\lambda_i = \lambda_* + \tau_i^{(0)}\big(1+\mathcal{O}(l^2\mathrm{e}^{-4l\sqrt{\nu_1-\lambda_*}})\big) + \mathcal{O}(\|\mathrm{A}_1(l)\|), \quad l\to+\infty. \tag{1.15}$$

*Here $\tau_i^{(0)} = \tau_i^{(0)}(l)$ are the roots of the polynomial $\det\big(\tau\mathrm{E}-\mathrm{A}_0(l)\big)$ taken counting multiplicity and ordered as follows*

$$0 \leqslant |\tau_1^{(0)}(l)| \leqslant |\tau_2^{(0)}(l)| \leqslant \ldots \leqslant |\tau_p^{(0)}(l)|.$$

*Each of these roots satisfies the estimate*

$$\tau_i^{(0)}(l) = \mathcal{O}(\|\mathrm{A}_0(l)\|), \quad l\to+\infty. \tag{1.16}$$

This theorem states that the leading terms of the asymptotics for the eigenvalues can be determined by that of the asymptotics for $A(\lambda_*, l)$. At the same time, the estimate for the error term in (1.15) can be worse than that in (1.8). In the following theorem we apply Theorem 1.5 to several important particular cases.

Let $\nu_2 > \nu_1$ be the second eigenvalue of the negative Dirichlet Laplacian in $\omega$, and $\phi_1 = \phi_1(x')$ be the eigenvalue associated with $\nu_1$ and normalized in $L_2(\omega)$. In the fifth section we will prove

**Lemma 1.1.** *Let the hypothesis of Theorem 1.4 hold true, and $p_\pm \geqslant 1$. Then the functions $\psi_i^\pm$ can be chosen so that*

$$\psi_1^\pm(x) = \beta_\pm \mathrm{e}^{\pm\sqrt{\nu_1-\lambda_*}x_1}\phi_1(x') + \mathcal{O}(\mathrm{e}^{\pm\sqrt{\nu_2-\lambda_*}x_1}), \quad \psi_i^\pm(x) = \mathcal{O}(\mathrm{e}^{\pm\sqrt{\nu_2-\lambda_*}x_1}), \tag{1.17}$$

*as $x_1 \to \mp\infty$, $i = 2, \ldots, p_\pm$, and the functions $\psi_i^\pm$ are orthonormalized in $L_2(\Pi)$.*

**Theorem 1.6.** *Let the hypothesis of Theorem 1.5 hold true, and $p_+ = 0$. Then the eigenvalues $\lambda_i$ satisfy the asymptotic formulas*

$$\begin{aligned}
&\lambda_i(l) = \lambda_* + \mathcal{O}\big(\mathrm{e}^{-2l(\sqrt{\nu_1-\lambda_*}+\sqrt{\nu_2-\lambda_*})}\big), \quad i = 1, \ldots, p-1, \\
&\lambda_p(l) = \lambda_* - 2\sqrt{\nu_1-\lambda_*}|\beta_-|^2\widetilde{\beta}_-\mathrm{e}^{-4l\sqrt{\nu_1-\lambda_*}} + \mathcal{O}\big(\mathrm{e}^{-2l(\sqrt{\nu_1-\lambda_*}+\sqrt{\nu_2-\lambda_*})} + l^2\mathrm{e}^{-6l\sqrt{\nu_1-\lambda_*}}\big),
\end{aligned} \tag{1.18}$$

*where the constant $\widetilde{\beta}_-$ is determined uniquely by the identity*

$$\begin{aligned}
&U_+(x) = \widetilde{\beta}_-\mathrm{e}^{-\sqrt{\nu_1-\lambda_*}x_1}\phi_1(x') + \mathcal{O}(\mathrm{e}^{\sqrt{\nu_2-\lambda_*}x_1}), \quad x_1 \to -\infty, \\
&U_+ := (\mathcal{H}_+ - \lambda_*)^{-1}\mathcal{L}_+\big(\mathrm{e}^{-\sqrt{\nu_1-\lambda_*}x_1}\phi_1(x')\big).
\end{aligned} \tag{1.19}$$

**Theorem 1.7.** *Let the hypothesis of Theorem 1.5 hold true, and $p_- = 0$. Then the eigenvalues $\lambda_i$ satisfy the asymptotic formulas*

$$\begin{aligned}
&\lambda_i(l) = \lambda_* + \mathcal{O}\big(\mathrm{e}^{-2l(\sqrt{\nu_1-\lambda_*}+\sqrt{\nu_2-\lambda_*})}\big), \quad i = 1, \ldots, p-1, \\
&\lambda_p(l) = \lambda_* - 2\sqrt{\nu_1-\lambda_*}|\beta_+|^2\widetilde{\beta}_+\mathrm{e}^{-4l\sqrt{\nu_1-\lambda_*}} + \mathcal{O}\big(\mathrm{e}^{-2l(\sqrt{\nu_1-\lambda_*}+\sqrt{\nu_2-\lambda_*})} + l^2\mathrm{e}^{-6l\sqrt{\nu_1-\lambda_*}}\big),
\end{aligned} \tag{1.20}$$

*where the constant $\widetilde{\beta}_+$ is uniquely determined by the identity*

$$\begin{aligned}
&U_+(x) = \widetilde{\beta}_+\mathrm{e}^{\sqrt{\nu_1-\lambda_*}x_1}\phi_1(x') + \mathcal{O}(\mathrm{e}^{\sqrt{\nu_2-\lambda_*}x_1}), \quad x_1 \to +\infty, \\
&U_+ := (\mathcal{H}_- - \lambda_*)^{-1}\mathcal{L}_+\big(\mathrm{e}^{-\sqrt{\nu_1-\lambda_*}x_1}\phi_1(x')\big)
\end{aligned}$$

These two theorems treat the first possible case when the number $\lambda_* \in \sigma$ is the eigenvalue of one of the operators $\mathcal{H}_\pm$ only. The formulas (1.18), (1.20) give the asymptotic expansion for the eigenvalue $\lambda_p$, and the asymptotic estimates for the other eigenvalues. At the same time, given arbitrary $\mathcal{L}_\pm$ and an eigenvalue $\lambda_*$ of $\mathcal{H}_\pm$, this eigenvalue is simple. In this case $p = 1$, and by Theorem 1.3 there exists the unique perturbed eigenvalue converging to $\lambda_*$, and Theorems 1.6, 1.7 provide its asymptotics.

**Theorem 1.8.** *Let the hypothesis of Theorem 1.4 hold true, and $p_\pm \geqslant 1$. Then the eigenvalues $\lambda_i$ satisfy the asymptotic formulas*

$$\begin{aligned}
&\lambda_i(l) = \lambda_* + \mathcal{O}(l\mathrm{e}^{-4l\sqrt{\nu_1-\lambda_*}}), \quad i = 1, \ldots, p-2,\\
&\lambda_{p-1}(l) = \lambda_* - 2|\beta_-\beta_+|\sqrt{\nu_1 - \lambda_*}\mathrm{e}^{-2l\sqrt{\nu_1-\lambda_*}} + \mathcal{O}(l\mathrm{e}^{-4l\sqrt{\nu_1-\lambda_*}}),\\
&\lambda_p(l) = \lambda_* + 2|\beta_-\beta_+|\sqrt{\nu_1 - \lambda_*}\mathrm{e}^{-2l\sqrt{\nu_1-\lambda_*}} + \mathcal{O}(l\mathrm{e}^{-4l\sqrt{\nu_1-\lambda_*}}),
\end{aligned} \tag{1.21}$$

*as $l \to +\infty$.*

This theorem deals with the second possible case when the number $\lambda_* \in \sigma$ is an eigenvalue of both operators $\mathcal{H}_\pm$. Similarly to Theorems 1.6, 1.7, the formulas (1.21) give the asymptotic expansions for $\lambda_{p-1}$ and $\lambda_p$, and the asymptotic estimates for the other eigenvalues. At the same time, the most possible case is that $\lambda_*$ is a simple eigenvalue of $\mathcal{H}_+$ and $\mathcal{H}_-$. In this case there exist only two perturbed eigenvalues converging to $\lambda_*$, and Theorem 1.8 give their asymptotic expansions.

Suppose now that under the hypothesis of Theorem 1.8 the inequality $\beta_-\beta_+ \neq 0$ holds true. In this case the leading terms of the asymptotic expansions for $\lambda_{p-1}$ and $\lambda_p$ have the same modulus but different signs. Moreover, these eigenvalues are simple. This situation is similar to what one has when dealing with a double-well problem with symmetric wells. We should also stress that in our case we assume no symmetry condition for the distant perturbations. It means that the mentioned phenomenon is the general situation and not the consequence of the symmetry. We also observe that the formulas (1.21) allow us to estimate the spectral gap between $\lambda_{p-1}$ and $\lambda_p$,

$$\lambda_2(l) - \lambda_1(l) = 4|\beta_-\beta_+|\sqrt{\nu_1 - \lambda_*}\mathrm{e}^{-2l\sqrt{\nu_1-\lambda_*}} + \mathcal{O}(l\mathrm{e}^{-4l\sqrt{\nu_1-\lambda_*}}), \quad l \to +\infty.$$

In conclusion we note that it is possible to calculate the asymptotic expansions for the eigenvalues $\lambda_i$, $i \leqslant p-1$, in Theorem 1.6, 1.7, and for $\lambda_i$, $i \leqslant p-2$, in Theorem 1.8. In order to do it, one should employ the technique of the proofs of the mentioned theorems and extract the next-to-leading term of the asymptotic for $A(\lambda_*, l)$. Then the obtained terms of this asymptotics should be treated as the matrix $A_0(l)$ in (1.14). We do not adduce such calculations in the article in order not to overload the text by quite bulky and technical details.

# 2 Proof of Theorem 1.1

Let $\Omega$ be a bounded non-empty subdomain of $\Pi$ defined as $\Omega := (-a, a) \times \omega$, where $a \in \mathbb{R}$, $a > 0$, $\gamma := (-a, a) \times \partial\omega$, and let $\mathcal{L}$ be an arbitrary bounded operator from $W^2_{2,0}(\Omega, \gamma)$ into $L_2(\Pi, \Omega)$. Assume that for all $u, u_1, u_2 \in W^2_{2,0}(\Omega, \gamma)$ the relations

$$(\mathcal{L}u_1, u_2)_{L_2(\Omega)} = (u_1, \mathcal{L}u_2)_{L_2(\Omega)}, \quad (\mathcal{L}u, u)_{L_2(\Omega)} \geqslant -c_0\|\nabla u\|^2_{L_2(\Omega)} - c_1\|u\|^2_{L_2(\Omega)} \tag{2.1}$$

hold true, where $c_0$, $c_1$ are constants, and $c_0$ obeys (1.3). As in the case of the operators $\mathcal{L}_\pm$, we can also regard the operator $\mathcal{L}$ as an unbounded one in $L_2(\Pi)$ with domain $W^2_{2,0}(\Pi)$. Suppose also that the operator $\mathcal{H}_\mathcal{L} := -\Delta^{(D)} + \mathcal{L}$ with domain $W^2_{2,0}(\Pi)$ is self-adjoint in $L_2(\Pi)$.

**Lemma 2.1.** $\sigma_{\mathrm{ess}}(\mathcal{H}_\mathcal{L}) = [\nu_1, +\infty)$.

*Proof.* We employ Weyl criterion to prove the theorem. Let $\lambda \in [\nu_1, +\infty)$ and let $\chi = \chi(t) \in C^\infty(\mathbb{R})$ be a cut-off function equalling one as $t < 0$, vanishing as $t > 1$, and

$$\int\limits_0^1 \chi^2(t)\,\mathrm{d}t = \frac{1}{2}.$$

We introduce a sequence of the functions

$$u_p(x) := \frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda-\nu_1}x_1}}{\sqrt{(2p+1)|\omega|}}\chi(|x_1| - p)\phi_1(x').$$

It is easy to make sure that

$$\|u_p\|_{L_2(\Pi)} = 1, \quad (u_p, \zeta)_{L_2(\Pi)} \xrightarrow[p\to+\infty]{} 0 \tag{2.2}$$

for each function $\zeta \in C^\infty(\Pi)$ vanishing on $\partial\Pi$ and having a compact support. Hence, $u_p$ converges to zero weakly in $L_2(\Pi)$ as $p \to +\infty$. Moreover, $\|u_p\|_{W^2_2(\Omega)} \to 0$ as $p \to +\infty$. It implies immediately that $\|\mathcal{L}u_p\|_{L_2(\Omega)} \to 0$ as $p \to +\infty$. Thus,

$$\|(-\Delta^{(D)} + \mathcal{L} - \lambda)u_p\|_{L_2(\Pi)} \to 0, \quad p \to +\infty.$$

By (2.2) it yields that $u_p$ is a singular sequence for $\mathcal{H}_\mathcal{L}$ at $\lambda$. Therefore, $[\nu_1, +\infty) \subseteq \sigma_{\mathrm{ess}}(\mathcal{H}_\mathcal{L})$. Let us prove the opposite inclusion.

Suppose $\lambda \in \sigma_{\mathrm{ess}}(\mathcal{H}_\mathcal{L})$. By Weyl criterion it follows that there exists a singular sequence $u_p$ for $\mathcal{H}_\mathcal{L}$ at $\lambda$. Employing $u_p$, we are going to construct a singular sequence for $-\Delta^{(D)}$ at $\lambda$. Due to the obvious identity $\sigma_{\mathrm{ess}}(-\Delta^{(D)}) = [\nu_1, +\infty)$, such fact will complete the proof.

Without loss of generality we assume that $\|u_p\|_{L_2(\Pi)} = 1$. Employing the inequality in (2.1) and denoting $f_p := \mathcal{H}_\mathcal{L}u_p$, we obtain a chain of relations,

$$\begin{aligned}
&(f_p, u_p)_{L_2(\Pi)} = (\mathcal{H}_\mathcal{L}u_p, u_p)_{L_2(\Pi)} = \|\nabla u_p\|^2_{L_2(\Pi)} - \lambda + (\mathcal{L}u_p, u_p)_{L_2(\Pi)},\\
&\|\nabla u_p\|^2_{L_2(\Pi)} = \lambda - (\mathcal{L}u_p, u_p)_{L_2(\Pi)} + (f_p, u_p)_{L_2(\Pi)} \leqslant \lambda + c_0\|\nabla u_p\|^2_{L_2(\Pi)} + c_1 + \|f_p\|_{L_2(\Pi)},\\
&(1 - c_0)\|\nabla u_p\|^2_{L_2(\Pi)} \leqslant \lambda + c_1 + \|f_p\|_{L_2(\Pi)} \leqslant C,
\end{aligned}$$

where the constant $C$ is independent of $p$. Therefore, $u_p$ is uniformly (in $p$) bounded in the norm of $W^1_2(\Pi)$.

Since the operator $\mathcal{H}_\mathcal{L}$ is self-adjoint, the inverse operator $(\mathcal{H}_\mathcal{L} - \mathrm{i})^{-1}$ exists and is bounded as an operator in $L_2(\Pi)$. The range of the inverse operator is

$W^2_{2,0}(\Pi)$ and we can thus regard it as an operator from $L_2(\Pi)$ into $W^2_{2,0}(\Pi)$. The operator $(\mathcal{H}_\mathcal{L} - \mathrm{i}) : W^2_{2,0}(\Pi) \to L_2(\Pi)$ is bounded. Therefore by Banach theorem on inverse operator (see, for instance, [16, Ch. 6, Sec. 23.1, Theorem 2]) the operator $(\mathcal{H}_\mathcal{L} - \mathrm{i})^{-1} : L_2(\Pi) \to W^2_{2,0}(\Pi)$ is bounded. This fact, the obvious identity $(\mathcal{H}_\mathcal{L} - \mathrm{i})u_p = (\lambda - \mathrm{i})u_p + f_p$ and the properties of $u_p$ and $f_p$ imply that the functions $u_p$ are bounded in $W^2_2(\Pi)$-norm uniformly in $p$. Extracting if needed a subsequence from $\{u_p\}$, we can assume that this sequence converges to zero weakly in $W^2_{2,0}(\widetilde{\Omega})$ and strongly in $W^1_2(\widetilde{\Omega})$, where $\widetilde{\Omega} := (-a-1, a+1) \times \omega$.

We denote $v_p(x) := \big(1 - \chi(|x_1| - a)\big)u_p(x)$. Clearly, $v_p = 0$ as $x \in \Omega$. Hence,

$$(-\Delta^{(D)} - \lambda)v_p = \widetilde{f}_p := f_p(1-\chi) + 2\nabla u_p \cdot \nabla\chi + u_p(\Delta + \lambda)\chi,$$

where $\chi = \chi(|x_1| - a)$, and in view of the established convergence for $u_p$,

$$\|v_p\|_{L_2(\Pi)} \to 1, \quad \|\widetilde{f}_p\|_{L_2(\Pi)} \to 0, \quad p \to +\infty.$$

It implies that $v_p$ is a singular sequence for $-\Delta^{(D)}$ at $\lambda$. $\square$

**Lemma 2.2.** *The discrete spectrum of $\mathcal{H}_\mathcal{L}$ consists of finitely many eigenvalues.*

*Proof.* Assume that the function $\chi$ introduced in the proof of Lemma 2.1 takes the values in $[0,1]$. Due to (2.1) we have

$$\begin{aligned}(u, \mathcal{H}_\mathcal{L}u)_{L_2(\Pi)} &\geqslant \|\nabla u\|^2_{L_2(\Pi)} - c_0\|\nabla u\|^2_{L_2(\Omega)} - c_1\|u\|^2_{L_2(\Omega)} \geqslant \\ &\geqslant \big(\nabla u, (1 - c_0\chi(|x_1| - a))\nabla u\big)_{L_2(\Pi)} - \big(u, c_1\chi(|x_1| - a)u\big)_{L_2(\Pi)}.\end{aligned} \tag{2.3}$$

We divide the tube $\Pi$ into three subsets $\overline{\Pi} = \overline{\Pi}^{(-)} \cup \overline{\Pi}^{(0)} \cup \overline{\Pi}^{(+)}$ defining them as $\Pi^{(-)} := (-\infty, -a-1) \times \omega$, $\Pi^{(+)} := (a+1, +\infty) \times \omega$, $\Pi^{(0)} := (-a-1, a+1) \times \omega$. By $\mathcal{H}^{(\pm)}_\mathcal{L}$ we indicate the Laplacian in $L_2(\Pi^{(\pm)})$ whose domain is the subset of the functions from $W^2_{2,0}(\Pi^{(\pm)}, \partial\Pi^{(\pm)} \setminus (\pm a \pm 1) \times \omega)$ satisfying Neumann boundary condition on $(\pm a \pm 1) \times \omega$. The symbol $\mathcal{H}^{(0)}_\mathcal{L}$ denotes the operator

$$-\operatorname{div}\big(1 - c_0\chi(|x_1| - a)\big)\nabla - c_1\chi(|x_1| - a)$$

in $L_2(\Pi^{(0)})$ with the domain formed by the functions from $W^2_{2,0}(\Pi^{(0)}, (-a-1, a+1) \times \partial\omega)$ satisfying Neumann condition on $\{-a-1\} \times \omega$ and $\{a+1\} \times \omega$. The inequality (2.3) implies that

$$\mathcal{H}_\mathcal{L} \geqslant \widehat{\mathcal{H}}_\mathcal{L} := \mathcal{H}^{(-)}_\mathcal{L} \oplus \mathcal{H}^{(0)}_\mathcal{L} \oplus \mathcal{H}^{(+)}_\mathcal{L}. \tag{2.4}$$

It is easy to see that $\mathcal{H}^{(\pm)}_\mathcal{L}$ are self-adjoint operators and $\sigma(\mathcal{H}^{(\pm)}_\mathcal{L}) = \sigma_{\mathrm{ess}}(\mathcal{H}^{(\pm)}_\mathcal{L}) = [\nu_1, +\infty)$. The self-adjoint operator $\mathcal{H}^{(0)}_\mathcal{L}$ is lower semibounded due to (1.3), and its spectrum is purely discrete. Moreover, the operator $\mathcal{H}^{(0)}_\mathcal{L}$ has finitely many eigenvalues in $(-\infty, \nu_1]$. Hence, the discrete spectrum of $\widehat{\mathcal{H}}_\mathcal{L}$ contains finitely many eigenvalues. Due to (2.4) and the minimax principle we can claim that the $k$-th eigenvalue of $\widehat{\mathcal{H}}_\mathcal{L}$ is estimated from above by the $k$-th eigenvalue of $\mathcal{H}_\mathcal{L}$. The former having finitely many discrete eigenvalues, the same is true for $\mathcal{H}_\mathcal{L}$. $\square$

The statement of Theorem 1.1 follows from Lemmas 2.1, 2.2, if one chooses $\mathcal{L} = \mathcal{L}_+$, $\Omega = \Omega_+$; $\mathcal{L} = \mathcal{L}_-$, $\Omega = \Omega_-$; $\mathcal{L} = \mathcal{L}_l$, $\Omega = \omega \times (-l - a_-, l + a_+)$.

## 3 Analysis of $\mathcal{H}_\pm$

In this section we establish certain properties of the operators $\mathcal{H}_\pm$ which will be employed in the proof of Theorems 1.2-1.7.

By $\mathbb{S}_\delta$ we indicate the set of all complex numbers separated from the half-line $[\nu_1, +\infty)$ by a distance greater than $\delta$. We choose $\delta$ so that $\sigma_{\mathrm{disc}}(\mathcal{H}_\pm) \subset \mathbb{S}_\delta$.

**Lemma 3.1.** *The operator $(\mathcal{H}_\pm - \lambda)^{-1} : L_2(\Pi) \to W_{2,0}^2(\Pi)$ is bounded and meromorphic w.r.t. $\lambda \in \mathbb{S}_\delta$. The poles of this operator are the eigenvalues of $\mathcal{H}_\pm$. For any $\lambda$ close to $p$-multiple eigenvalue $\lambda_*$ of $\mathcal{H}_\pm$ the representation*

$$(\mathcal{H}_\pm - \lambda)^{-1} = -\sum_{j=1}^{p} \frac{\psi_j^\pm(\cdot, \psi_j^\pm)_{L_2(\Pi)}}{\lambda - \lambda_*} + \mathcal{T}_4^\pm(\lambda) \tag{3.1}$$

*holds true. Here $\psi_j^\pm$ are the eigenfunctions associated with $\lambda^\pm$ and orthonormalized in $L_2(\Pi)$, while $\mathcal{T}_4^\pm(\lambda) : L_2(\Pi) \to W_{2,0}^2(\Pi)$ is a bounded operator being holomorphic w.r.t. $\lambda$ in a small neighbourhood of $\lambda^\pm$. The relations*

$$(\mathcal{T}_4^\pm(\lambda) f, \psi_j^\pm)_{L_2(\Pi)} = 0, \quad j = 1, \ldots, p, \tag{3.2}$$

*are valid.*

*Proof.* According to the results of [14, Ch. V, Sec. 3.5], the operator $(\mathcal{H}_\pm - \lambda)^{-1}$ considered as an operator $L_2(\Pi)$ is bounded and meromorphic w.r.t. $\lambda \in \mathbb{S}_\delta$ and its poles are the eigenvalues of $\mathcal{H}_\pm$. It is not difficult to check that

$$(\mathcal{H}_\pm - \lambda - \eta)^{-1} - (\mathcal{H}_\pm - \lambda)^{-1} = \eta(\mathcal{H}_\pm - \lambda)^{-1}(\mathrm{I} - \eta(\mathcal{H}_\pm - \lambda)^{-1})^{-1}(\mathcal{H}_\pm - \lambda)^{-1}.$$

This identity implies that the operator $(\mathcal{H}_\pm - \lambda)^{-1}$ is also bounded and meromorphic w.r.t. $\lambda \in \mathbb{S}_\delta$ as an operator into $W_{2,0}^2(\Pi)$, and its poles are the eigenvalues of $\mathcal{H}_\pm$. Consider $\lambda$ ranging in a small neighbourhood of $\lambda_*$. The formula (3.21) in [14, Ch. V, Sec. 3.5] gives rise to the representation (3.1), where the operator $\mathcal{T}_4^\pm(\lambda) : L_2(\Pi) \to L_2(\Pi)$ is bounded and holomorphic w.r.t. $\lambda$. The self-adjointness of $\mathcal{H}_\pm$ and (3.1) yield that for any $f \in L_2(\Pi)$ the identities

$$\mathcal{T}_4^\pm(\lambda) f = (\mathcal{H}_\pm - \lambda)^{-1} \widetilde{f}, \quad \widetilde{f} = f - \sum_{j=1}^{p} \psi_j^\pm (f, \psi_j^\pm)_{L_2(\Pi)}, \tag{3.3}$$

and (3.2) hold true. Let $\Psi^\perp$ be a subspace of the functions in $L_2(\Pi)$ which are orthogonal to $\psi_j^\pm$, $j = m, \ldots, m+p-1$. The identities (3.3) mean that

$$\mathcal{T}_4^\pm(\lambda)\big|_{\Psi^\perp} = (\mathcal{H}_\pm - \lambda)^{-1}\big|_{\Psi^\perp}. \tag{3.4}$$

The operator $(\mathcal{H}_\pm - \lambda)^{-1}\big|_{\Psi^\perp}$ is holomorphic w.r.t. $\lambda$ as an operator from $\Psi^\perp$ into $W_{2,0}^2(\Pi) \cap \Psi^\perp$. This fact is due to holomorphy in $\lambda$ of the operator $(\mathcal{H}_\pm - \lambda) : W_{2,0}^2(\Pi) \cap \Psi^\perp \to \Psi^\perp$ and the invertibility of this operator (see [14, Ch. VII,

Sec. 1.1]). Therefore, the restriction of $\mathcal{T}_4^\pm(\lambda)$ on $\Psi^\perp$ is holomorphic w.r.t. $\lambda$ as an operator into $W_{2,0}^2(\Pi)$. Taking into account (3.3), we conclude that for any $f \in L_2(\Pi)$ the function $\mathcal{T}_4^\pm(\lambda)f$ is holomorphic w.r.t. $\lambda$. The holomorphy in a weak sense implies the holomorphy in the norm sense [14, Ch. VII, Sec. 1.1], and we arrive at the statement of the lemma. ☐

Let $0 < \nu_1 < \nu_2 \leqslant \ldots \leqslant \nu_j \leqslant \ldots$ be the eigenvalues of the negative Dirichlet Laplacian in $\omega$ taken in a non-decreasing order counting multiplicity, and $\phi_i = \phi_i(x')$ be the associated eigenfunctions orthonormalized in $L_2(\omega)$. We denote $\Pi_a^\pm := \Pi \cap \{x : \pm x_1 > \pm a\}$.

**Lemma 3.2.** *Suppose that $u \in W_2^1(\Pi_a^\pm)$ is a solution to the boundary value problem*

$$(\Delta + \lambda)u = 0, \quad x \in \Pi_a^\pm, \qquad u = 0, \quad x \in \partial\Pi \cap \overline{\Pi}_a^\pm,$$

*where $\lambda \in \mathbb{S}_\delta$. Then the function $u$ can be represented as*

$$u(x) = \sum_{j=1}^{\infty} \alpha_j \mathrm{e}^{-s_j(\lambda)(\pm x_1 - a)} \phi_j(x'), \quad \alpha_j = \int\limits_{\omega} u(a, x')\phi_j(x')\,\mathrm{d}x'. \tag{3.5}$$

*The series (3.5) converges in $W_2^p(\Pi_b^\pm)$ for any $\Pi_b^\pm \subset \Pi_a^\pm$, $p \geqslant 0$. The coefficients $\alpha_j$ satisfy the identity*

$$\sum_{j=1}^{\infty} |\alpha_j|^2 = \|u(\cdot, a)\|_{L_2(\omega)}^2. \tag{3.6}$$

*Proof.* In view of the obvious change of variables it is sufficient to prove the lemma for $\Pi_0^+$. It is clear that $v \in C^\infty(\overline{\Pi}_0^+ \setminus \omega_0)$, where $\omega_0 := \{0\} \times \overline{\omega}$. Since $v \in W_2^1(\Pi_0^+)$ and $v(x, \lambda) = 0$ as $x \in \partial\Pi_0^+ \setminus \omega_0$, the representation

$$u(x, \lambda) = \sum_{j=1}^{\infty} \phi_j(x') \int\limits_{\omega} u(x_1, t, \lambda)\phi_j(t)\,\mathrm{d}t$$

holds true for any $x_1 \geqslant 0$ in $L_2(\omega)$. We have the identity

$$\sum_{j=1}^{\infty} \left| \int\limits_{\omega} u(x, \lambda)\phi_j(x')\,\mathrm{d}x' \right|^2 = \|u(x_1, \cdot, \lambda)\|_{L_2(\omega)}^2 \tag{3.7}$$

for any $x_1 \geqslant 0$. Employing the equation for $u$ we obtain

$$\begin{aligned}
&\frac{d^2}{dx_1^2} \int\limits_{\omega} u(x, \lambda)\phi_j(x')\,\mathrm{d}x' = -\int\limits_{\omega} \phi_j(x')\,(\Delta_{x'} + \lambda)\,u(x, \lambda)\,\mathrm{d}x' \\
&= -\int\limits_{\omega} u(x, \lambda)\,(\Delta_{x'} + \lambda)\,\phi_j(x')\,\mathrm{d}x' = (\nu_j - \lambda)\int\limits_{\omega} u(x, \lambda)\phi_j(x')\,\mathrm{d}x', \quad x_1 > 0.
\end{aligned}$$

The function $v$ belonging to $W_2^1(\Pi_0^+)$, the relation obtained implies that

$$\int\limits_{\omega} u(x,\lambda)\phi_j(x')\,\mathrm{d}x' = \alpha_j \mathrm{e}^{-\sqrt{\nu_j-\lambda}x_1}. \tag{3.8}$$

We have employed here the identity

$$\lim_{x_1\to+0}\int\limits_0^\pi u(x,\lambda)\phi_j(x')\,\mathrm{d}x' = \int\limits_\omega u(0,x',\lambda)\phi_j(x')\,\mathrm{d}x',$$

which follows from the estimate

$$\begin{aligned}&\left|\int\limits_\omega \big(u(x,\lambda)-u(0,x',\lambda)\big)\phi_j(x')\,\mathrm{d}x'\right|\\ &\qquad\leqslant \left|\int\limits_\omega\int\limits_0^{x_1}\frac{\partial u}{\partial t_1}(t,x')\phi_j(x')\,\mathrm{d}t_1\,\mathrm{d}x'\right| \leqslant \sqrt{|x_1|}\left\|\frac{\partial u}{\partial x_1}\right\|_{L_2(\Pi_0^+)}.\end{aligned}$$

The identities (3.7), (3.8) yield (3.6). For $x_1 \geqslant b > 0$ the coefficients of (3.5) decays exponentially as $j \to +\infty$, that implies the convergence of the series in (3.5) in $W_2^p(\Pi_b^+)$, $p \geqslant 0$. ☐

For $l \geqslant a_- + a_+$ we introduce the operators $\mathcal{T}_6^\pm(\lambda,l) : L_2(\Pi,\Omega_\mp) \to L_2(\Pi,\Omega_\pm)$,

$$\mathcal{T}_6^\pm(\lambda,l) := \mathcal{L}_\pm\mathcal{S}(\pm 2l)(\mathcal{H}_\mp-\lambda)^{-1}. \tag{3.9}$$

**Lemma 3.3.** *The operator $\mathcal{T}_6^\pm$ is bounded and meromorphic w.r.t. $\lambda\in\mathbb{S}_\delta$. For any compact set $\mathbb{K}\subset\mathbb{S}_\delta$ separated from $\sigma_{\mathrm{disc}}(\mathcal{H}_\mp)$ by a positive distance the estimates*

$$\left\|\frac{\partial^i\mathcal{T}_6^\pm}{\partial\lambda^i}\right\| \leqslant Cl^i\mathrm{e}^{-2l\operatorname{Re}s_1(\lambda)},\quad i=0,1,\lambda\in\mathbb{K}, \tag{3.10}$$

*hold true, where the constant $C$ is independent of $\lambda\in\mathbb{K}$ and $l\geqslant a_-+a_+$. For any $\lambda$ close to a $p$-multiple eigenvalue $\lambda_*$ of $\mathcal{H}_\mp$ the representation*

$$\mathcal{T}_6^\pm(\lambda,l) = -\sum_{j=1}^p\frac{\varphi_j^\mp(\cdot,\psi_j^\mp)_{L_2(\Omega_\mp)}}{\lambda-\lambda_*}+\mathcal{T}_7^\pm(\lambda,l),\quad \varphi_j^\mp := \mathcal{L}_\pm\mathcal{S}(\pm 2l)\psi_j^\mp, \tag{3.11}$$

*is valid. Here $\psi_j^\mp$ are the eigenfunctions associated with $\lambda_*$ and orthonormalized in $L_2(\Pi)$, while the operator $\mathcal{T}_7^\pm(\lambda,l) : L_2(\Pi,\Omega_\mp)\to L_2(\Pi,\Omega_\pm)$ is bounded and holomorphic w.r.t. $\lambda$ close to $\lambda_*$ and satisfies the estimates*

$$\left\|\frac{\partial^i\mathcal{T}_7^\pm}{\partial\lambda^i}\right\| \leqslant Cl^{i+1}\mathrm{e}^{-2l\operatorname{Re}s_1(\lambda)},\quad i=0,1, \tag{3.12}$$

*where the constant $C$ is independent of $\lambda$ close to $\lambda_*$ and $l\geqslant a_-+a_+$. The identities*

$$\mathcal{T}_7^\pm(\lambda_*,l) = \mathcal{L}_\pm\mathcal{S}(\pm 2l)\mathcal{T}_4^\mp(\lambda_*) \tag{3.13}$$

*hold true.*

*Proof.* We prove the lemma for $\mathcal{T}_6^+$ only; the proof for $\mathcal{T}_6^-$ is similar. Let $f \in L_2(\Pi, \Omega_-)$, and denote $u := (\mathcal{H}_- - \lambda)^{-1} f$. The function $f$ having compact support, by Lemma 3.2 the function $u$ can be represented as the series (3.5) for $x_1 \geqslant a_-$. Hence,

$$\big(\mathcal{S}(2l)u\big)(x) = \sum_{j=1}^{\infty} \alpha_j \mathrm{e}^{-2s_j(\lambda)l} \mathrm{e}^{-s_j(\lambda)(\pm x_1 - a)} \phi_j(x').$$

Employing this representation and (3.6), we obtain

$$\begin{aligned}
&\left\| \sum_{j=1}^{\infty} \alpha_j \mathrm{e}^{-2s_j(\lambda)l} \mathrm{e}^{-s_j(\lambda)(x_1 - a_-)} \phi_j(x') \right\|_{W_2^2(\Omega_+)} \leqslant \\
&\leqslant C \sum_{j=1}^{\infty} |\alpha_j| \mathrm{e}^{-2l \operatorname{Re} s_j(\lambda)} \|\mathrm{e}^{-s_j(\lambda)(\cdot - a_-)}\|_{W_2^2(-a_+, a_+)} \big( \|\Delta_{x'} \phi_j\|_{L_2(\omega)} + \|\phi_j\|_{L_2(\omega)} \big) \leqslant \\
&\leqslant C \sum_{j=1}^{\infty} |\alpha_j| |s_j(\lambda)|^{3/2} \nu_j \mathrm{e}^{-(2l - a_- - a_+) \operatorname{Re} s_j(\lambda)} \leqslant \\
&\leqslant C \left( \sum_{j=1}^{\infty} |\alpha_j|^2 \right)^{1/2} \left( \sum_{j=1}^{\infty} (\nu_j^{7/2} + |\lambda|^{7/2}) \mathrm{e}^{-2(2l - a_- - a_+) \operatorname{Re} s_j(\lambda)} \right)^{1/2} \leqslant \\
&\leqslant C(1 + |\lambda|^{7/4}) \mathrm{e}^{-(2l - a_- - a_+) \operatorname{Re} s_1(\lambda)} \|u(a, \cdot)\|_{L_2(\omega)},
\end{aligned} \tag{3.14}$$

where the constant $C$ is independent of $\lambda \in \mathbb{S}_\delta$ and $l \geqslant a_- + a_+$. Here we have also applied the well-known estimate [18, Ch. IV, Sec. 1.5, Theorem 5]

$$c j^{\frac{2}{n-1}} \leqslant \nu_j \leqslant C j^{\frac{2}{n-1}}. \tag{3.15}$$

By direct calculations we check that

$$\left( \frac{\partial \mathcal{S}(2l)u}{\partial \lambda} \right)(x, \lambda, l) = \sum_{j=1}^{\infty} \frac{\alpha_j}{2s_j(\lambda)} (x_1 - a_- + 2l) \mathrm{e}^{-2s_j(\lambda)l} \mathrm{e}^{-s_j(\lambda)(x_1 - a_-)} \phi_j(x').$$

Proceeding in the same way as in (3.14) we obtain

$$\left\| \frac{\partial \mathcal{S}(2l)u}{\partial \lambda} \right\|_{W_2^2(\Omega_+)} \leqslant Cl(1 + |\lambda|^{5/4}) \mathrm{e}^{-(2l - a_- - a_+) \operatorname{Re} s_1(\lambda)} \|u(a, \cdot)\|_{L_2(\omega)}, \tag{3.16}$$

where the constant $C$ is independent of $\lambda \in \mathbb{S}_\delta$ and $l \geqslant a_- + a_+$. In the same way we check that

$$\left\| \frac{\partial^2 \mathcal{S}(2l)u}{\partial \lambda^2} \right\|_{W_2^2(\Omega_+)} \leqslant Cl(1 + |\lambda|^{3/4}) \mathrm{e}^{-(2l - a_- - a_+) \operatorname{Re} s_1(\lambda)} \|u(a, \cdot)\|_{L_2(\omega)}, \tag{3.17}$$

where the constant $C$ is independent of $\lambda \in \mathbb{S}_\delta$ and $l \geqslant a_- + a_+$. Lemma 3.1 implies

$$u(a_-, \cdot) = -\sum_{j=1}^{p} \frac{\psi_j^-(a_-, \cdot)(f, \psi_j^-)_{L_2(\Pi)}}{\lambda - \lambda_*} + \big(\mathcal{T}_4^-(\lambda) f\big)(a_-, \cdot).$$

This representation and (3.6), (3.14), (3.16), (3.17) lead us to the statement of the lemma. $\square$

# 4 Reduction of the eigenvalue equation for $\mathcal{H}_l$

In this section we reduce the eigenvalue equation

$$\mathcal{H}_l\psi = \lambda\psi \tag{4.1}$$

to an operator equation in the space $L_2(\Omega_-) \oplus L_2(\Omega_+)$. The reduction will be one of the key ingredients in the proofs of Theorems 1.2-1.7. Hereafter we assume that $l \geqslant a_- + a_+$.

Let $f_\pm \in L_2(\Pi, \Omega_\pm)$ be a pair of arbitrary functions, and let the functions $u_\pm$ satisfy the equations

$$(\mathcal{H}_\pm - \lambda)u_\pm = f_\pm, \tag{4.2}$$

where $\lambda \in \mathbb{S}_\delta$. We choose $\delta$ so that $\sigma_{\mathrm{disc}}(\mathcal{H}_\pm) \subset \mathbb{S}_\delta$. We construct a solution to (4.1) as

$$\psi = \mathcal{S}(l)u_- + \mathcal{S}(-l)u_+. \tag{4.3}$$

Suppose that the function $\psi$ defined in this way satisfies (4.1). We substitute (4.3) into (4.1) to obtain

$$0 = (\mathcal{H}_l - \lambda)\psi = (\mathcal{H}_l - \lambda)\mathcal{S}(l)u_- + (\mathcal{H}_l - \lambda)\mathcal{S}(-l)u_+.$$

By direct calculations we check that

$$\begin{aligned}
(\mathcal{H}_l - \lambda)\mathcal{S}(l)u_- &= \mathcal{S}(l)(-\Delta^{(D)} + \mathcal{L}_- - \lambda)\mathcal{S}(-l)\mathcal{S}(l)u_- + \mathcal{S}(-l)\mathcal{L}_+\mathcal{S}(2l)u_- = \\
&= \mathcal{S}(l)f_- + \mathcal{S}(-l)\mathcal{L}_+\mathcal{S}(2l)u_-, \\
(\mathcal{H}_l - \lambda)\mathcal{S}(-l)u_+ &= \mathcal{S}(-l)f_+ + \mathcal{S}(l)\mathcal{L}_-\mathcal{S}(-2l)u_+.
\end{aligned}$$

Hence,

$$\mathcal{S}(l)\big(f_- + \mathcal{L}_-\mathcal{S}(-2l)u_+\big) + \mathcal{S}(-l)\big(f_+ + \mathcal{L}_+\mathcal{S}(2l)u_-\big) = 0.$$

Since the functions $f_\pm + \mathcal{L}_\pm\mathcal{S}(\pm 2l)u_\mp$ are compactly supported, it follows that the functions $\mathcal{S}(\mp l)\big(f_\pm + \mathcal{L}_\pm\mathcal{S}(\pm 2l)u_\mp\big)$ are compactly supported, too, and their supports are disjoint. Thus, the last equation obtained is equivalent to the pair of the equations

$$f_- + \mathcal{L}_-\mathcal{S}(-2l)u_+ = 0, \quad f_+ + \mathcal{L}_+\mathcal{S}(2l)u_- = 0. \tag{4.4}$$

These equations are equivalent to the equation (4.1). The proof of this fact is the subject of

**Lemma 4.1.** *To any solution* $\boldsymbol{f} := (f_-, f_+) \in L_2(\Omega_-) \oplus L_2(\Omega_+)$ *of (4.4) and functions* $u_\pm$ *solving (4.2) there exists a unique solution of (4.1) given by (4.2), (4.3). For any solution* $\psi$ *of (4.1) there exists a unique* $\boldsymbol{f} \in L_2(\Omega_-) \oplus L_2(\Omega_+)$ *solving (4.4) and the unique functions* $u_\pm$ *satisfying (4.2) such that* $\psi$ *is given by (4.2), (4.3). The equivalence holds for any* $\lambda \in \mathbb{S}_\delta$.

*Proof.* It was shown above that if $\boldsymbol{f} \in L_2(\Omega_-) \oplus L_2(\Omega_+)$ solves (4.4) and the functions $u_\pm$ are the solutions to (4.2), the function $\psi$ defined by (4.3) solves (4.1).

Suppose that $\psi$ is a solution of (4.1). This functions satisfies the equation

$$(-\Delta - \lambda)\psi = 0, \qquad -l + a_- < x_1 < l - a_+, \quad x' \in \omega, \tag{4.5}$$

and vanishes as $-l + a_- < x_1 < l - a_+$, $x' \in \partial\omega$. Due to standard smoothness improving theorems (see, for instance, [18, Ch. Ⅳ, Sec. 2.2]) it implies that $\psi \in C^\infty(\{x : -l + a_- < x_1 < l - a_+, \quad x' \in \overline{\omega}\})$. Hence, the numbers

$$\rho_j^\pm = \rho_j^\pm(l) := \frac{1}{2}\int\limits_\omega \left(\psi(0, x', l) \pm \frac{1}{s_j(\lambda)}\frac{\partial \psi}{\partial x_1}(0, x', l)\right)\phi_j(x')\,\mathrm{d}x'$$

are well-defined. Employing the equation (4.5), the identity $\psi = 0$ as $x \in \partial\Pi$, and the smoothness of $\psi$ we integrate by parts,

$$\begin{aligned}
\rho_j^\pm &= -\frac{1}{2\nu_j}\int\limits_\omega \left(\psi(0, x', l) \pm \frac{1}{s_j(\lambda)}\frac{\partial \psi}{\partial x_1}(0, x', l)\right)\Delta_{x'}\phi_j(x')\,\mathrm{d}x' = \\
&= -\frac{1}{2\nu_j}\int\limits_\omega \phi_j(x')\Delta_{x'}\left(\psi(0, x', l) \pm \frac{1}{s_j(\lambda)}\frac{\partial \psi}{\partial x_1}(0, x', l)\right)\mathrm{d}x' = \\
&= \frac{1}{2\nu_j}\int\limits_\omega \phi_j(x')\left(\frac{\partial^2}{\partial x_1^2} + \lambda\right)\left(\psi \pm \frac{1}{s_j(\lambda)}\frac{\partial \psi}{\partial x_1}\right)\Bigg|_{x_1=0}\mathrm{d}x' = \\
&= \frac{1}{2\nu_j^p}\int\limits_\omega \phi_j(x')\left(\frac{\partial^2}{\partial x_1^2} + \lambda\right)^p\left(\psi \pm \frac{1}{s_j(\lambda)}\frac{\partial \psi}{\partial x_1}\right)\Bigg|_{x_1=0}\mathrm{d}x'
\end{aligned}$$

for any $p \in \mathbb{N}$. In view of (3.15) it yields that $\sum\limits_{j=1}^{\infty} j^p|\rho_j^\pm| < \infty$ for any $p \in \mathbb{N}$. Now we introduce the functions $u_\pm$,

$$\begin{aligned}
u_\pm(x_1 \mp l, x', l) &:= \sum_{j=1}^\infty \rho_j^\pm(l)\mathrm{e}^{\pm s_j(\lambda)x_1}\phi_j(x'), && x \in \Pi_0^\mp, \\
u_\pm(x_1 \mp l, x', l) &:= \psi(x, l) - u_\mp(x_1 \pm l, x', l), && x \in \Pi_0^\pm,
\end{aligned}$$

and conclude that

$$u_-(x_1 + l, x', l), u_+(x_1 - l, x', l) \in C^\infty(\overline{\Pi_0^\pm}) \cap W_{2,0}^2(\Pi_0^\pm, \partial\Pi \cap \partial\Pi_0^\pm).$$

The smoothness of $\psi$ gives rise to the representations

$$\psi(0,x',l)=\sum_{j=1}^{\infty}\big(\rho_j^+(l)+\rho_j^-(l)\big)\phi_j(x'),$$
$$\frac{\partial\psi}{\partial x_1}(0,x',l)=\sum_{j=1}^{\infty}\big(\rho_j^+(l)-\rho_j^-(l)\big)s_j(\lambda)\phi_j(x').$$

These relations and the aforementioned smoothness of $u_\pm$ imply that $u_\pm\in W^2_{2,0}(\Pi)$. We also infer that the introduced functions $u_\pm$ satisfy (4.3). We define now the vector $\boldsymbol{f}=(f_-,f_+)\in L_2(\Omega_-)\oplus L_2(\Omega_+)$ by $f_-:=-\mathcal{L}_-u_+(x_1-2l,x',l)$, $f_+:=-\mathcal{L}_+u_-(x_1+2l,x',l)$. Let us check that the functions $u_\pm$ satisfy (4.2); it will imply that the vector $\boldsymbol{f}$ just introduced solves (4.6).

The definition of $u_\pm$ implies that

$$(\Delta+\lambda)u_\pm(x_1\mp l,x',l)=0,\quad x\in\overline{\Pi}_0^{\mp}.$$

Thus,

$$(\Delta+\lambda)u_-=0,\quad x\in\overline{\Pi}_l^{+},\qquad (\Delta+\lambda)u_+=0,\quad x\in\overline{\Pi}_{-l}^{-}.$$

By these equations, the equation for $\psi$, and the definitions of $u_-(x_1+l,x',l)$ for $x\in\Pi_0^-$, we obtain that for $x\in\Pi_l^-$

$$\begin{aligned}(-\Delta-\lambda+\mathcal{L}_-)u_-(x,l)&=(-\Delta-\lambda+\mathcal{L}_-)\big(\psi(x_1-l,x',l)-u_+(x_1-2l,x',l)\big)=\\ &=-(-\Delta-\lambda+\mathcal{L}_-)u_+(x_1-2l,x',l)=-\mathcal{L}_-u_+(x_1-2l,x',l)=f_-(x).\end{aligned}$$

Therefore, $(\mathcal{H}_--\lambda)u_-=f_-$. The relation $(\mathcal{H}_+-\lambda)u_+=f_+$ can be established in the same way. $\square$

Suppose that $\lambda\in\mathbb{S}_\delta\setminus\sigma_*$. In this case $u_\pm=(\mathcal{H}_\pm-\lambda)^{-1}f_\pm$. This fact together with the definition (3.9) of $\mathcal{T}_6^\pm$ implies that $\mathcal{L}_\pm\mathcal{S}(\pm 2l)u_\mp=\mathcal{T}_6^\pm(\lambda,l)f_\pm$. Substituting the identity obtained into (4.4), we arrive at the equation

$$\boldsymbol{f}+\mathcal{T}_2(\lambda,l)\boldsymbol{f}=0,\tag{4.6}$$

where $\boldsymbol{f}:=(f_-,f_+)\in L_2(\Omega_-)\oplus L_2(\Omega_+)$, where the operator $\mathcal{T}_2:L_2(\Omega_-)\oplus L_2(\Omega_+)\to L_2(\Omega_-)\oplus L_2(\Omega_+)$ is defined in (1.4).

*Proof of Theorem 1.2.* The inequality (2.3) yields that the operator $\mathcal{H}_l$ is lower semibounded with lower bound $-c_1$. Together with Theorem 1.1 it implies that the discrete eigenvalues of the operator are located in $[-c_1,\nu_1)$. We introduce the set $\mathbb{K}_\delta:=[-c_1,\nu_1-\delta)\setminus\bigcup\limits_{\lambda\in\sigma_*}(\lambda-\delta,\lambda+\delta)$. It satisfies the hypothesis of Lemma 3.3, and the estimate (3.10) implies that

$$\|\mathcal{T}_2(\lambda,l)\|\leqslant C(\delta)\mathrm{e}^{-2l\operatorname{Re}s_1(\lambda)},$$

if $l$ is large enough and $\lambda \in \mathbb{K}_\delta$. The constant $C$ in this estimate is independent of $\lambda \in \mathbb{K}_\delta$ and $l$ large enough. Hence, for such $\lambda$ and $l$ the operator $(\mathrm{I} + \mathcal{T}_2(\lambda, l))$ is boundedly invertible. Therefore, the equation (4.6) has no nontrivial solutions, if $l$ is large enough and $\lambda \in \mathbb{K}_\delta$. Since $\mathbb{K}_\delta \cap \sigma_* = \emptyset$, the identity $\boldsymbol{f} = 0$ implies that $u_\pm = (\mathcal{H}_\pm - \lambda)^{-1} f_\pm = 0$, i.e., $\psi = 0$. Thus, the equation (4.1) has no nontrivial solution for $\lambda \in \mathbb{K}_\delta$, if $l$ is large enough. Therefore, $\mathbb{K}_\delta \cap \sigma_{\mathrm{disc}}(\mathcal{H}_l) = \emptyset$, if $l$ is large enough. The number $\delta$ being arbitrary, the last identity completes the proof. ☐

Assume that $\lambda_* \in \sigma_*$ is $(p_- + p_+)$-multiple. Lemma 3.3 implies that for $\lambda$ close to $\lambda_*$ the representation (1.5) holds true, where the operator $\mathcal{T}_3$ is given by

$$
\begin{aligned}
&\mathcal{T}_3(\lambda, l)\boldsymbol{f} := (\mathcal{T}_7^-(\lambda, l) f_+, \mathcal{T}_6^+(\lambda, l) f_-), \quad \text{if} \quad p_- = 0,\\
&\mathcal{T}_3(\lambda, l)\boldsymbol{f} := (\mathcal{T}_6^-(\lambda, l) f_+, \mathcal{T}_7^+(\lambda, l) f_-), \quad \text{if} \quad p_+ = 0,\\
&\mathcal{T}_3(\lambda, l)\boldsymbol{f} := (\mathcal{T}_7^-(\lambda, l) f_+, \mathcal{T}_7^+(\lambda, l) f_-), \quad \text{if} \quad p_- \neq 0, \quad p_+ \neq 0.
\end{aligned}
$$

Lemma 3.3 yields also that the operator $\mathcal{T}_3(\lambda, l)$ is bounded and holomorphic w.r.t. $\lambda$ in a small neighbourhood of $\lambda_*$, and satisfies the estimate

$$
\left\| \frac{\partial^i \mathcal{T}_3}{\partial \lambda^i} \right\| \leqslant C l^{i+1} \mathrm{e}^{-2l \operatorname{Re} s_1(\lambda)}, \quad i = 0, 1, \tag{4.7}
$$

where the constant $C$ is independent of $l \geqslant a_- + a_+$ and $\lambda$ close to $\lambda_*$.

Suppose that $\lambda \neq \lambda_*$ is an eigenvalue of $\mathcal{H}_l$ converging to $\lambda_*$. In this case the identity $\boldsymbol{f} = 0$ leads us to the relations $u_\pm = (\mathcal{H}_\pm - \lambda)^{-1} f_\pm = 0$, $\psi = 0$. Thus, the corresponding equation (4.6) has a nontrivial solution. Let us solve this equation.

We substitute (1.5) into (4.6), and arrive at the following equation:

$$
\boldsymbol{f} - \frac{1}{\lambda - \lambda_*} \sum_{i=1}^{p} \boldsymbol{\phi}_i \mathcal{T}_1^{(i)} \boldsymbol{f} + \mathcal{T}_3 \boldsymbol{f} = 0. \tag{4.8}
$$

In view of the estimate (4.7) the operator $(\mathrm{I} + \mathcal{T}_3(\lambda, l))$ is invertible, and the operator $(\mathrm{I} + \mathcal{T}_3)^{-1}$ is bounded and holomorphic w.r.t. $\lambda$ in a small neighbourhood of $\lambda_*$. Applying this operator to the last equation gives rise to one more equation,

$$
\boldsymbol{f} = \frac{1}{\lambda - \lambda_*} \sum_{i=1}^{p} \boldsymbol{\Phi}_i \mathcal{T}_1^{(i)} \boldsymbol{f}, \tag{4.9}
$$

where $\boldsymbol{\Phi}_i = \boldsymbol{\Phi}_i(\cdot, \lambda, l) := (\mathrm{I} + \mathcal{T}_3(\lambda, l))^{-1} \boldsymbol{\phi}_i(\cdot, l)$. We denote $k_i = k_i(\lambda, l) := (\lambda - \lambda_*)^{-1} \mathcal{T}_1^{(i)} \boldsymbol{f}$, and apply the functionals $\mathcal{T}_1^{(j)}$, $j = 1, \ldots, p$, to the equation (4.9) that leads us to a system of linear equations (1.12), where $\boldsymbol{k} := (k_1 \ldots k_p)^t$.

Given a non-trivial solution of (4.6), the associated vector $\boldsymbol{k}$ is non-zero, since otherwise the definition of $k_i$, and (4.9) would imply that $\boldsymbol{f} = 0$. Therefore, if $\lambda \neq \lambda_*$ is an eigenvalue of $\mathcal{H}_l$, the system (1.12) has a non-trivial solution. It is true, if and only if the equation (1.7) holds true. Thus, each eigenvalue of $\mathcal{H}_l$ converging to $\lambda_*$ and not coinciding with $\lambda_*$ should satisfy this equation.

Let us show that if $\lambda_*$ is an eigenvalue of $\mathcal{H}_l$, it satisfies (1.7) as well. In this case $\lambda_*$ the associated eigenfunction is given by (4.3) that is due to Lemma 4.1. The self-adjointness of $\mathcal{H}_\pm$ implies that

$$(f_\pm, \psi_i^\pm)_{L_2(\Omega_\pm)} = \big((\mathcal{H}_\pm - \lambda_*)u_\pm, \psi_i^\pm\big)_{L_2(\Omega_\pm)} = \big(u_\pm, (\mathcal{H}_\pm - \lambda_*)\psi_i^\pm\big)_{L_2(\Omega_\pm)} = 0, \quad (4.10)$$

$i = 1, \ldots, p_\pm$. Therefore, the functions $u_\pm$ can be represented as

$$u_-(\cdot, l) = \mathcal{T}_4^-(\lambda_*)f_- - \sum_{i=1}^{p_-} k_i\psi_i^-, \quad u_+(\cdot, l) = \mathcal{T}_4^+(\lambda_*)f_+ - \sum_{i=1}^{p_+} k_{i+p_-}\psi_i^+, \quad (4.11)$$

where $k_i$ are numbers to be found. Employing the relation (3.13) and substituting (4.11) into (4.4), we obtain the equation

$$\boldsymbol{f} + \mathcal{T}_3(\lambda_*, l)\boldsymbol{f} = \sum_{i=1}^{p} k_i\boldsymbol{\phi}_i(\cdot, l),$$

which is equivalent to

$$\boldsymbol{f} = \sum_{i=1}^{p} k_i\boldsymbol{\Phi}_i(\cdot, \lambda_*, l). \quad (4.12)$$

The relations (4.10) can be rewritten as $\mathcal{T}_1^{(i)}\boldsymbol{f} = 0$, $i = 1, \ldots, p$, that together with (4.12) implies the system (1.12) for $\lambda = \lambda_*$. The vector $\boldsymbol{k}$ is non-zero since otherwise the relations (4.12), (4.11) would imply $\boldsymbol{f} = 0$, $u_\pm = 0$, $\psi = 0$. Thus, $\det \mathrm{A}(\lambda_*, l) = 0$, which coincides with the equation (1.7) for $\lambda = \lambda_*$.

Let $\lambda$ be a root of (1.7), converging to $\lambda_*$ as $l \to +\infty$. We are going to prove that in this case the equation (4.1) has a non-trivial solution, i.e., $\lambda$ is an eigenvalue of $\mathcal{H}_l$. The equation (4.1) being satisfied, it follows that the system (1.12) has a nontrivial solution $\boldsymbol{k}$. We specify this solution by the requirement

$$\|\boldsymbol{k}\|_{\mathbb{C}^p} = 1, \quad (4.13)$$

and define $\boldsymbol{f} := \sum\limits_{i=1}^{p} k_i\boldsymbol{\Phi}_i(\cdot, \lambda, l) \in \boldsymbol{f} \in L_2(\Omega_-) \oplus L_2(\Omega_+)$. The system (1.12) and the definition of $\mathcal{T}_1^{(i)}$ give rise to the identities

$$\begin{aligned} &(f_-, \psi_i^-)_{L_2(\Omega_-)} = \mathcal{T}_1^{(i)}\boldsymbol{f} = (\lambda - \lambda_*)k_i, && i = 1, \ldots, p_-, \\ &(f_+, \psi_i^+)_{L_2(\Omega_-)} = \mathcal{T}_1^{(i+p_-)}\boldsymbol{f} = (\lambda - \lambda_*)k_{i+p_-}, && i = 1, \ldots, p_+. \end{aligned} \quad (4.14)$$

Taking these identities into account and employing (3.1), in the case $\lambda \neq \lambda_*$ we arrive at the formulas

$$u_- = -\sum_{i=1}^{p_-} k_i\psi_i^- + \mathcal{T}_4^-(\lambda)f_-, \quad u_+ = -\sum_{i=1}^{p_+} k_{i+p_-}\psi_i^+ + \mathcal{T}_4^+(\lambda)f_+. \quad (4.15)$$

In the case $\lambda = \lambda_*$ we adopt these formulas as the definition of the functions $u_\pm$ that is possible due to the identities (4.14) with $\lambda = \lambda_*$ and (3.4).

If $\lambda \neq \lambda_*$, we employ the system (1.12) to check by direct calculations that $\boldsymbol{f}$ solves (4.9), and thus the equations (4.4). If $\lambda = \lambda_*$, the identity (3.13) implies

$$\begin{aligned}\big(\mathcal{L}_-\mathcal{S}(-2l)u_+, \mathcal{L}_+\mathcal{S}(2l)u_-\big) = \mathcal{T}_3(\lambda_*, l)\boldsymbol{f} - \sum_{i=1}^{p} k_i \boldsymbol{\phi}_i(\cdot, l) = \mathcal{T}_3(\lambda_*, l)\boldsymbol{f} \\ - \sum_{i=1}^{p} k_i (\mathrm{I} + \mathcal{T}_3(\lambda_*, l))\boldsymbol{\Phi}_i(\cdot, \lambda_*, l) = \mathcal{T}_3(\lambda_*, l)\boldsymbol{f} - (\mathrm{I} + \mathcal{T}_3(\lambda_*, l))\boldsymbol{f} = -\boldsymbol{f}.\end{aligned}$$

Hence, the equation (4.4) holds true. With Lemma 4.1 in mind we therefore conclude that the function $\psi$ defined by (4.3) solves (4.1).

Let us prove that $\psi \not\equiv 0$; it will imply that $\lambda$ is an eigenvalue of $\mathcal{H}_l$. Lemma 1.1 implies that the functions $\varphi_i^\pm$ obey the estimate

$$\|\varphi_i^\pm\|_{L_2(\Omega_\mp)} \leqslant C\mathrm{e}^{-2ls_1(\lambda_*)}, \tag{4.16}$$

where the constant $C$ is independent of $l$. This estimate together with (4.7) gives rise to the similar estimates for $\boldsymbol{\Phi}_i$:

$$\begin{aligned}&\|\boldsymbol{\Phi}_i\|_{L_2(\Omega_-)\oplus L_2(\Omega_+)} \leqslant C\mathrm{e}^{-2ls_1(\lambda_*)}, \\ &\left\|\frac{\partial \boldsymbol{\Phi}_i}{\partial \lambda}(\cdot, \lambda, l)\right\|_{L_2(\Omega_-)\oplus L_2(\Omega_+)} \leqslant Cl^2 \mathrm{e}^{-2l(s_1(\lambda_*)+s_1(\lambda))},\end{aligned} \tag{4.17}$$

where the constant $C$ is independent of $\lambda$ and $l$. The latter inequality is based on the formula

$$\frac{\partial}{\partial \lambda}(\mathrm{I} + \mathcal{T}_3)^{-1} = -(\mathrm{I} + \mathcal{T}_3)^{-1}\frac{\partial \mathcal{T}_3}{\partial \lambda}(\mathrm{I} + \mathcal{T}_3)^{-1},$$

which follows from the obvious identities:

$$\mathrm{I} = (\mathrm{I} + \mathcal{T}_3)(\mathrm{I} + \mathcal{T}_3)^{-1}, \quad 0 = \frac{\partial \mathcal{T}_3}{\partial \lambda}(\mathrm{I} + \mathcal{T}_3)^{-1} + (\mathrm{I} + \mathcal{T}_3)\frac{\partial}{\partial \lambda}(\mathrm{I} + \mathcal{T}_3)^{-1}.$$

Hence, the asymptotics (1.11) is valid. The vector $\boldsymbol{k}$ being non-zero, it implies $\psi \not\equiv 0$. We summarize the results of the section in

**Lemma 4.2.** *The eigenvalues of $\mathcal{H}_l$ converging to a $(p_- + p_+)$-multiple $\lambda_* \in \sigma_*$ coincide with the roots of (1.7) converging to $\lambda_*$. The associated eigenfunctions are given by (4.3), (4.11), (4.15), where the coefficients $k_i$ form non-trivial solutions to (1.12). If $\lambda(l)$ is an eigenvalue of $\mathcal{H}_l$ converging to $\lambda_*$ as $l \to +\infty$, its multiplicity coincides with the number of linear independent solutions of the system (1.12) taken for $\lambda = \lambda(l)$. The associated eigenfunctions satisfy (1.11).*

# 5 Proof of Theorem 1.3

Throughout this and next sections the parameter $\lambda$ is assumed to belong to a small neighbourhood of $\lambda_*$, while $l$ is supposed to be large enough. We begin with the proof of Lemma 1.1.

*Proof of Lemma 1.1.* We will prove the lemma for $\psi_i^-$ only, the case of $\psi_i^+$ is completely similar. According to Lemma 3.2 the functions $\psi_i^-$ can be represented as the series (3.5) in $\Pi_{a_-}^+$. Let $\Psi$ be the space of the $L_2(\Pi)$-functions spanned over $\psi_i^-$. Each function from this space satisfies the representation (3.5) in $\Pi_{a_-}^+$. We introduce two quadratic forms in this finite-dimensional space, the first being generated by the inner product in $L_2(\Pi)$, while the other is defined as $\mathfrak{q}(u,v) := \alpha_1[u]\overline{\alpha_1[v]}$, where the $\alpha_1[u]$, $\alpha_1[v]$ are the first coefficients in the representations (3.5) for $u$ and $v$ in $\Pi_{a_-}^+$. By the theorem on simultaneous diagonalization of two quadratic forms, we conclude that we can choose the basis in $\Psi$ so that both these forms are diagonalized. Denoting this basis as $\psi_i^-$, we conclude that these functions are orthonormalized in $L_2(\Pi)$, and

$$\mathfrak{q}(\psi_i^-, \psi_j^-) = 0, \quad \text{if} \quad i \neq j. \tag{5.1}$$

Suppose that for all the functions $\psi_i^-$ the coefficient $\alpha_1[\psi_i^-]$ is zero. In this case we arrive at (1.17), where $\beta_- = 0$. If at least one of the functions $\psi_i^-$ has a nonzero coefficient $\alpha_1$, say $\psi_1^-$, the identity (5.1) implies that $\alpha_1[\psi_i^-] = 0$, $i \geqslant 2$, and we arrive again at (1.17). $\square$

The definition of $\mathrm{A}_{ij}$ and (4.17) imply the estimates

$$|A_{ij}(\lambda,l)| \leqslant C\mathrm{e}^{-2ls_1(\lambda_*)}, \qquad \left|\frac{\partial A_{ij}}{\partial\lambda}(\lambda,l)\right| \leqslant Cl^2\mathrm{e}^{-2l(s_1(\lambda_*)+s_1(\lambda))}, \tag{5.2}$$

where the constant $C$ is independent of $\lambda$ and $l$. Moreover, the holomorphy of the operator $\mathcal{T}_3$ w.r.t. $\lambda$ yields that the functions $A_{ij}$ are holomorphic w.r.t. $\lambda$. This fact and the estimate (5.2) allow us to claim that the right hand side of (1.7) reads as follows,

$$F(\lambda,l) := \det\big((\lambda-\lambda_*)\mathrm{E} + \mathrm{A}(\lambda,l)\big) = (\lambda-\lambda_*)^p + \sum_{i=0}^{p-1} P_i(\lambda,l)(\lambda-\lambda_*)^i,$$

where the functions $P_i$ are holomorphic w.r.t. $\lambda$, and obey the estimate

$$|P_i(\lambda,l)| \leqslant C\mathrm{e}^{-2(p-i)ls_1(\lambda_*)} \tag{5.3}$$

with the constant $C$ independent of $\lambda$ and $l$. Given $\delta > 0$, this estimate implies

$$\left|\sum_{i=0}^{p-1} P_i(\lambda,l)(\lambda-\lambda_*)^i\right| < |\lambda-\lambda_*|^p \quad \text{as} \quad |\lambda-\lambda_*| = \delta,$$

if $l$ is large enough. Now we employ Rouche theorem to infer that the function $F(\lambda, l)$ has the same amount of the zeroes inside the disk $\{\lambda : |\lambda - \lambda_*| < \delta\}$ as the function $\lambda \mapsto (\lambda - \lambda_*)^p$ does. Thus, the function $F(\lambda, l)$ has exactly $p$ zeroes in this disk (counting their orders), if $l$ is large enough. The number $\delta$ being arbitrary, we conclude that the equation (1.7) has exactly $p$ roots (counting their orders) converging to $\lambda_*$ as $l \to +\infty$.

**Lemma 5.1.** *Suppose that $\lambda_1(\lambda)$ and $\lambda_2(\lambda)$ are different roots of (1.7), and $\boldsymbol{k}_1(l)$ and $\boldsymbol{k}_2(l)$ are the associated non-trivial solutions to (1.12) normalized by (4.13). Then*

$$\big(\boldsymbol{k}_1(l), \boldsymbol{k}_2(l)\big)_{\mathbb{C}^p} = \mathcal{O}(l\mathrm{e}^{-2ls_1(\lambda_*)}), \quad l \to +\infty.$$

*Proof.* According to Lemma 4.2, the numbers $\lambda_1(l) \neq \lambda_2(l)$ are the eigenvalues of $\mathcal{H}_l$, and the associated eigenfunctions $\psi_i(x, l)$, $i = 1, 2$, are generated by (4.3), (4.11), (4.15), where $k_i$ are components of the vectors $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, respectively.

Using the representations (3.5) for $\psi_i^+$ in $\Pi^{\pm}_{\pm a_+}$ and for $\psi_i^-$ in $\Pi^{\pm}_{\pm a_-}$, by direct calculations one can check that

$$\big(\psi_i^+, \mathcal{S}(2l)\psi_j^-\big)_{L_2(\Pi)} = \mathcal{O}(l\mathrm{e}^{-2ls_1(\lambda_*)}), \quad l \to +\infty.$$

The operator being self-adjoint, the eigenfunctions $\psi_i(x, l)$ are orthogonal in $L_2(\Pi)$. Now be Lemma 4.2 and the last identity we obtain

$$0 = (\psi_1, \psi_2)_{L_2(\Pi)} = \sum_{i=1}^{p} k_i^{(1)} \overline{k_i^{(2)}} + \mathcal{O}(\mathrm{e}^{-2ls_1(\lambda_*)}), \quad l \to +\infty,$$

that completes the proof. $\square$

For each root $\lambda(l) \xrightarrow[l \to +\infty]{} \lambda_*$ of (1.7) the system (1.12) has a finite number of linear independent solutions. Without loss of generality we assume that these solutions are orthonormalized in $\mathbb{C}^p$. We consider the set of all such solutions associated with all roots of (1.7) converging to $\lambda_*$ as $l \to +\infty$, and indicate these vectors as $\boldsymbol{k}_i = \boldsymbol{k}_i(l)$, $i = 1, \ldots, q$. In view of the assumption for $\boldsymbol{k}_i$ just made and Lemma 5.1 the vectors $\boldsymbol{k}_i$ satisfy (1.13).

For the sake of brevity we denote $\mathrm{B}(\lambda, l) := (\lambda - \lambda_*)\mathrm{E} - \mathrm{A}(\lambda, l)$.

**Lemma 5.2.** *Let $\lambda(l) \xrightarrow[l \to +\infty]{} \lambda_*$ be a root of (1.7) and $\boldsymbol{k}_i$, $i = N, \ldots, N+m$, $m \geqslant 0$, be the associated solutions to (1.12). Then for any $\boldsymbol{h} \in \mathbb{C}^p$ the representation*

$$\mathrm{B}^{-1}(\lambda, l)\boldsymbol{h} = \sum_{i=N}^{N+m} \frac{\mathcal{T}_8^{(i)}(l)\boldsymbol{h}}{\lambda - \lambda(l)} \boldsymbol{k}_i(l) + \mathcal{T}_9(\lambda, l)\boldsymbol{h}$$

*holds true for all $\lambda$ close to $\lambda(l)$. Here $\mathcal{T}_8^{(i)} : \mathbb{C}^p \to \mathbb{C}$ are functionals, while the matrix $\mathcal{T}_9(\lambda, l)$ is holomorphic w.r.t. $\lambda$ in a neighbourhood of $\lambda(l)$.*

*Proof.* The matrix B being holomorphic w.r.t. $\lambda$, the inverse $\mathrm{B}^{-1}$ is meromorphic w.r.t. $\lambda$ and has a pole at $\lambda(l)$. The residue at this pole is a linear combination of $\boldsymbol{k}_i$, and for any $\boldsymbol{h}\in\mathbb{C}^p$ we have

$$\mathrm{B}^{-1}(\lambda,l)\boldsymbol{h}=\frac{1}{(\lambda-\lambda(l))^s}\sum_{i=N}^{N+m}\boldsymbol{k}_i\mathcal{T}_8^{(i)}(l)\boldsymbol{h}+\mathcal{O}\big((\lambda-\lambda(l))^{-s+1}\big),\quad \lambda\to\lambda(l),\tag{5.4}$$

where $s\geqslant 1$ is the order of the pole, and $\mathcal{T}_8^{(i)}:\mathbb{C}^p\to\mathbb{C}$ are some functionals.

We are going to prove that $s=1$. Let $g_\pm=g_\pm(x)\in L_2(\Pi,\Omega_\pm)$ be arbitrary functions. We consider the equation

$$(\mathcal{H}_l-\lambda)u=g:=\mathcal{S}(-l)g_++\mathcal{S}(l)g_-,\tag{5.5}$$

where $\lambda$ is close to $\lambda(l)$ and $\lambda\neq\lambda(l)$, $\lambda\neq\lambda_*$. The results of [14, Ch. VII, Sec. 3.5] imply that for such $\lambda$ the function $u$ can be represented as

$$u=-\frac{1}{\lambda-\lambda(l)}\sum_{i=N}^{N+m}(g,\psi_i)_{L_2(\Pi)}\psi_i+\mathcal{T}_{10}(\lambda,l)g,\tag{5.6}$$

where $\psi_i$ are the eigenfunctions of $\mathcal{H}_l$ associated with $\lambda(l)$ and orthonormalized in $L_2(\Pi)$, while the operator $\mathcal{T}_{10}(\lambda,l)$ is bounded and holomorphic w.r.t. $\lambda$ close to $\lambda(l)$ as an operator in $L_2(\Pi)$. The eigenvalue $\lambda(l)$ of the operator $\mathcal{H}_l$ is $(m+1)$-multiple by the assumption and Lemma 4.2. Completely by analogy with the proof of Lemma 4.1 one can make sure that the equation (5.5) is equivalent to

$$\boldsymbol{f}+\mathcal{T}_2(\lambda,l)\boldsymbol{f}=\boldsymbol{g},\tag{5.7}$$

where $\boldsymbol{g}:=(g_-,g_+)\in L_2(\Omega_-)\oplus L_2(\Omega_+)$, and the solution of (5.5) is given by

$$u=\mathcal{S}(l)u_-+\mathcal{S}(-l)u_+,\quad u_\pm=(\mathcal{H}_\pm-\lambda)^{-1}f_\pm.\tag{5.8}$$

Proceeding as in (4.6), (4.8), (4.9), one can solve (5.7),

$$\boldsymbol{f}=\sum_{i=1}^{p}U_i\boldsymbol{\Phi}_i+\widetilde{\boldsymbol{f}},\quad \boldsymbol{U}=\mathrm{B}^{-1}(\lambda,l)\boldsymbol{h},\quad \widetilde{\boldsymbol{f}}:=(\mathrm{I}+\mathcal{T}_3)^{-1}\boldsymbol{g},\tag{5.9}$$

where $U_i:=\mathcal{T}_1^{(i)}\boldsymbol{f}$, and the vector $\boldsymbol{U}:=(U_1\ldots U_p)^t$ is a solution to

$$\mathrm{B}(\lambda,l)\boldsymbol{U}=\boldsymbol{h},\tag{5.10}$$

$$\boldsymbol{h}=(h_1\ldots h_p)^t,\quad h_i:=\mathcal{T}_1^{(i)}(\lambda)(\mathrm{I}+\mathcal{T}_3(\lambda,l))^{-1}\boldsymbol{g}\tag{5.11}$$

Knowing the vectors $\boldsymbol{U}$ and $\boldsymbol{f}$, we can restore the functions $u_\pm$ by (3.1),

$$u_-(\cdot,\lambda,l)=-\sum_{i=1}^{p_-}U_i(\lambda,l)\psi_i^- -\sum_{i=1}^{p}U_i(\lambda,l)\mathcal{T}_4^-(\lambda,l)\Phi_i^-+\mathcal{T}_4^-(\lambda)\widetilde{f}_-,$$

$$u_+(\cdot,\lambda,l) = -\sum_{i=1}^{p_+} U_{i+p_-}(\lambda,l)\psi_i^+ - \sum_{i=1}^{p} U_i(\lambda,l)\mathcal{T}_4^+(\lambda,l)\Phi_i^+ + \mathcal{T}_4^+(\lambda)\widetilde{f}_+,$$

where $\Phi_i^\pm$ and $\widetilde{f}_\pm$ are introduced as $\boldsymbol{\Phi}_i = (\Phi_i^-, \Phi_i^+)$, $\widetilde{\boldsymbol{f}} = (\widetilde{f}_-, \widetilde{f}_+)$. In these formulas we have also employed the system (5.10) in the following way:

$$(f_-, \psi_j^-)_{L_2(\Pi)} = \mathcal{T}_1^{(j)}\boldsymbol{f} = h_j + \sum_{i=1}^{p} A_{ji}U_i = (\lambda-\lambda_*)U_j, \quad j = 1,\ldots,p_-,$$

$$(f_+, \psi_j^+)_{L_2(\Pi)} = \mathcal{T}_1^{(j+p_-)}\boldsymbol{f} = h_{j+p_-} + \sum_{i=1}^{p} A_{ji}U_i = (\lambda-\lambda_*)U_{j+p_-}, \quad j = 1,\ldots,p_+.$$

The estimates (4.17) allow us to infer that

$$\|\mathcal{T}_4^\pm(\lambda,l)\Phi_i^\pm\|_{L_2(\Pi)} = \mathcal{O}(\mathrm{e}^{-2ls_1(\lambda_*)}),$$

while in view of holomorphy of $\mathcal{T}_4^\pm$ and (4.7) we have

$$\|\mathcal{T}_4^\pm(\lambda)\widetilde{f}_\pm\|_{L_2(\Pi)} \leqslant C\|\boldsymbol{g}\|_{L_2(\Omega_-)\oplus L_2(\Omega_+)},$$

where the constant $C$ is independent of $\lambda$. Now we use the first of the relations (5.8) and obtain

$$\begin{aligned} u(\cdot,\lambda,l) = &-\sum_{i=1}^{p_-} U_i(\lambda,l)\Big(\mathcal{S}(l)\psi_i^- + \mathcal{O}(\mathrm{e}^{-2ls_1(\lambda_*)})\Big) \\ &-\sum_{i=1}^{p_+} U_{i+p_-}(\lambda,l)\Big(\mathcal{S}(-l)\psi_i^+ + \mathcal{O}(\mathrm{e}^{-2ls_1(\lambda_*)})\Big) + \mathcal{O}\left(\|\boldsymbol{g}\|_{L_2(\Omega_-)\oplus L_2(\Omega_+)}\right). \end{aligned} \tag{5.12}$$

Now we compare the representations (5.6) and (5.12) and conclude that the coefficients $U_i(\lambda,l)$, $i=1,\ldots,p$ has a simple pole at $\lambda(l)$. By (5.9) it implies that the vector $\mathrm{B}^{-1}(\lambda,l)\boldsymbol{h}$ has a simple pole at $\lambda(l)$, where $\boldsymbol{h}$ is defined by (5.11). It follows from (4.7) and (5.11) that for each $\boldsymbol{h}\in\mathbb{C}^p$ there exists $\boldsymbol{g}\in L_2(\Omega_-)\oplus L_2(\Omega_+)$ so that $\boldsymbol{h}$ is given by (5.11). Together with (5.4) it completes the proof. ☐

**Lemma 5.3.** *The number $\lambda(l) \xrightarrow[l\to+\infty]{} \lambda_*$ is a $m$-th order zero of $F(\lambda,l)$ if and only if it is a $m$-multiple eigenvalue of $\mathcal{H}_l$.*

*Proof.* Let $\lambda^{(i)}(l) \xrightarrow[l\to+\infty]{} \lambda_*$, $i=1,\ldots,M$, be the different zeroes of $F(\lambda,l)$, and $r_i$, $i=1,\ldots,M$, be the orders of these zeroes. By Lemma 4.2, each zero $\lambda^{(i)}(l)$ is an eigenvalue of $\mathcal{H}_l$; its multiplicity will be indicated as $m_i\geqslant 1$. To prove the lemma it is sufficient to show that $m_i = r_i$, $i=1,\ldots,M$.

Let us prove first that $m_i\leqslant r_i$. In accordance with Lemma 4.2 the multiplicity $m_i$ coincides with a number of linear independent solutions of (1.12) with $\lambda = \lambda^{(i)}(l)$. Hence,

$$\operatorname{rank}\mathrm{B}(\lambda(l),l) = p - m_i. \tag{5.13}$$

By the assumption

$$\frac{\partial^j}{\partial\lambda^j}\det \mathrm{B}(\lambda,l)=0,\quad j=1,\ldots,r_i-1,\qquad \frac{\partial^{r_i}}{\partial\lambda^{r_i}}\det \mathrm{B}(\lambda,l)\neq 0,$$

as $\lambda=\lambda^{(i)}(l)$. Let $B_j=B_j(\lambda,l)$ be the columns of the matrix B, i.e., $\mathrm{B}=(B_1,\ldots,B_p)$. Employing the well-known formula

$$\frac{\partial}{\partial\lambda}\det\mathrm{B}=\det\left(\frac{\partial B_1}{\partial\lambda}\,B_2\,\ldots\,B_p\right)+\det\left(B_1\,\frac{\partial B_2}{\partial\lambda}\,\ldots\,B_p\right)+\det\left(B_1\,B_2\,\ldots\,\frac{\partial B_p}{\partial\lambda}\right),$$

one can check easily that for each $0\leqslant j\leqslant m_i-1$

$$\frac{\partial^j}{\partial\lambda^j}\det\mathrm{B}(\lambda,l)\Big|_{\lambda=\lambda^{(i)}(l)}=\sum_{\varsigma}c_\varsigma\det\mathrm{B}_\varsigma,$$

where $c_\varsigma$ are constants, and at least $(p-m_i+1)$ columns of each matrix $\mathrm{B}_\varsigma$ are those of B. In view of (5.13) these columns are linear dependent, and therefore $\det\mathrm{B}_\varsigma=0$ for each $\varsigma$. Thus $m_i-1\geqslant r_i-1$ that implies the desired inequality.

Now it is sufficient to check that $q=\sum\limits_{i=1}^{M}m_i=\sum\limits_{i=1}^{M}r_i=p$ to prove that $m_i=r_i$. Lemma 5.2 yields that for a given fixed $\delta$ small enough and $l$ large enough

$$\mathrm{B}^{-1}(\lambda,l)\boldsymbol{h}=\sum_{i=1}^{q}\frac{\mathcal{T}_8^{(i)}(l)\boldsymbol{h}}{\lambda-\lambda^{(i)}(l)}\boldsymbol{k}_i(l)+\mathcal{T}_9(\lambda,l)\boldsymbol{h},\tag{5.14}$$

for any $\boldsymbol{h}\in\mathbb{C}^p$ and $\lambda$ so that $|\lambda-\lambda_*|=\delta$. It is also assumed that $|\lambda^{(i)}(l)-\lambda_*|<\delta$ for the considered values of $l$. We integrate this identity to obtain

$$\frac{1}{2\pi\mathrm{i}}\int\limits_{|\lambda-\lambda_*|=\delta}\mathrm{B}^{-1}(\lambda,l_s)\boldsymbol{h}\,\mathrm{d}\lambda=\sum_{i=1}^{q}\boldsymbol{k}_i\mathcal{T}_8^{(i)}(l)\boldsymbol{h}.\tag{5.15}$$

Due to (5.2) we conclude that

$$\frac{1}{2\pi\mathrm{i}}\int\limits_{|\lambda-\lambda_*|=\delta}\mathrm{B}^{-1}(\lambda,l)\,\mathrm{d}\lambda\xrightarrow[l\to+\infty]{}\frac{1}{2\pi\mathrm{i}}\int\limits_{|\lambda-\lambda_*|=\delta}\frac{\mathrm{E}\,\mathrm{d}\lambda}{\lambda-\lambda_*}=\mathrm{E}.\tag{5.16}$$

Hence the right hand side of (5.15) converges to $\boldsymbol{h}$. By (1.13) it implies that $q=p$ for $l$ large enough. $\square$

The statement of Theorem 1.3 follows from the proven lemma.

## 6 Asymptotics for the eigenelements of $\mathcal{H}_l$

In this section we prove Theorems 1.4-1.7. Throughout the section the hypothesis of Theorem 1.4 is assumed to hold true.

Theorem 1.3 implies that the number of the vectors $\boldsymbol{k}_i$ introduced in the previous section equals $p$. Let $\mathrm{S} = \mathrm{S}(l)$ be the matrix with columns $\boldsymbol{k}_i(l)$, $i = 1, \ldots, p$, i.e., $\mathrm{S}(l) = (\boldsymbol{k}_1(l) \ldots \boldsymbol{k}_p(l))$. Without loss of generality we can assume that $\det \mathrm{S}(l) \geqslant 0$.

**Lemma 6.1.** $\det \mathrm{S}(l) = 1 + \mathcal{O}(l\mathrm{e}^{-2ls_1(\lambda_*)})$, *as* $l \to +\infty$.

*Proof.* The relations (1.13) yield $\mathrm{S}^2(l) = \mathrm{E} + \mathcal{O}(l\mathrm{e}^{-2ls_1(\lambda_*)})$, $l \to +\infty$, that implies $\det^2 \mathrm{S}(l) = 1 + \mathcal{O}(l\mathrm{e}^{-2ls_1(\lambda_*)})$. The last identity proves the lemma. □

Lemma 6.1 implies that there exists the inverse matrix $\mathrm{S}^{-1}(l)$ for $l$ large enough.

**Lemma 6.2.** *The matrix* $\mathrm{R}(\lambda, l) := \mathrm{S}^{-1}(l)\mathrm{A}(\lambda, l)\mathrm{S}(l)$ *reads as follows*

$$\mathrm{R} = \begin{pmatrix} \lambda_1 - \lambda_* + (\lambda - \lambda_1)r_{11} & (\lambda - \lambda_2)r_{12} & \ldots & (\lambda - \lambda_p)r_{1p} \\ (\lambda - \lambda_1)r_{21} & \lambda_2 - \lambda_* + (\lambda - \lambda_2)r_{22} & \ldots & (\lambda - \lambda_p)r_{2p} \\ \vdots & \vdots & & \vdots \\ (\lambda - \lambda_1)r_{p1} & (\lambda - \lambda_2)r_{p2} & \ldots & \lambda_p - \lambda_* + (\lambda - \lambda_p)r_{pp} \end{pmatrix}.$$

*where* $\lambda_i = \lambda_i(l)$, *while the functions* $r_{ij} = r_{ij}(\lambda, l)$ *are holomorphic w.r.t.* $\lambda$ *close to* $\lambda_*$ *and obey the uniform in* $\lambda$ *and* $l$ *estimates*

$$|r_{ij}(\lambda, l)| \leqslant Cl^2\mathrm{e}^{-2l(s_1(\lambda_*)+s_1(\lambda))}. \tag{6.1}$$

*Proof.* The system (1.12) implies

$$\begin{aligned} \mathrm{A}(\lambda, l)\boldsymbol{k}_i &= \mathrm{A}(\lambda_i(l), l)\boldsymbol{k}_i + \big(\mathrm{A}(\lambda, l) - \mathrm{A}(\lambda_i(l), l)\big)\boldsymbol{k}_i \\ &= (\lambda_i(l) - \lambda_*)\boldsymbol{k}_i + \big(\mathrm{A}(\lambda, l) - \mathrm{A}(\lambda_i(l), l)\big)\boldsymbol{k}_i. \end{aligned}$$

The matrix $\mathrm{A}(\lambda, l) - \mathrm{A}(\lambda_i(l), l)$ is holomorphic w.r.t. $\lambda$, and

$$\widetilde{\mathrm{A}}(\lambda, \lambda_i(l), l) := \mathrm{A}(\lambda, l) - \mathrm{A}(\lambda_i(l), l) = \int\limits_{\lambda_i(l)}^{\lambda} \frac{\partial \mathrm{A}}{\partial \lambda}(z, l)\,\mathrm{d}z. \tag{6.2}$$

Due to (5.2) and (1.12) the last identity implies that

$$\mathrm{A}(\lambda, l)\boldsymbol{k}_i = (\lambda_i(l) - \lambda_*)\boldsymbol{k}_i + (\lambda - \lambda_i)\boldsymbol{K}_i(\lambda, l), \tag{6.3}$$

where the vectors $\boldsymbol{K}_i(\lambda, l)$ are holomorphic w.r.t. $\lambda$ close to $\lambda_*$ and satisfy the uniform in $\lambda$ and $l$ estimate

$$\|\boldsymbol{K}_i\|_{\mathbb{C}^p} \leqslant Cl^2\mathrm{e}^{-2l(s_1(\lambda_*)+s_1(\lambda))}.$$

By Lemma 5.3 the vectors $\boldsymbol{k}_i$, $i=1,\ldots,p$, form a basis in $\mathbb{C}^p$. Hence,

$$\boldsymbol{K}_i(\lambda,l)=\sum_{j=1}^{p} r_{ij}(\lambda,l)\boldsymbol{k}_j(l),\quad \big(r_{i1}(\lambda,l)\ldots r_{ip}(\lambda,l)\big)^t=\mathrm{S}^{-1}(l)\boldsymbol{K}_i(\lambda,l). \tag{6.4}$$

Due to Lemma 6.1, the relations (1.13), and the established properties of $\boldsymbol{K}_i$ we infer that the functions $r_{ij}$ are holomorphic w.r.t. $\lambda$ and satisfy (6.1). Taking into account (6.3) and (6.4), we arrive at the statement of the lemma. □

**Lemma 6.3.** *The polynomial* $\det\big(\tau\mathrm{E}-\mathrm{A}(\lambda_*,l)\big)$ *has exactly $p$ roots $\tau_i=\tau_i(l)$, $i=1,\ldots,p$ counting multiplicity which satisfy (1.9).*

*Proof.* It is clear that $\det\big(\tau\mathrm{E}-\mathrm{A}(\lambda_*,l)\big)$ is a polynomial of $p$-th order, this is why it has $p$ roots $\tau_i(l)$, $i=1,\ldots,p$, counting multiplicity. It is easy to check that

$$\det\big(\tau\mathrm{E}-\mathrm{A}(\lambda_*,l)\big)=\tau^p+\sum_{j=0}^{p-1}P_j(\lambda_*,l)\tau^j,$$

where the functions $P_j(\lambda_*,l)$ satisfy (5.3). We make a change of variable $\tau_i=z_i\mathrm{e}^{-2ls_1(\lambda_*)}$, and together with the representation just obtained it leads us to the equation for $z_i$,

$$z^p+\sum_{j=0}^{p-1}\mathrm{e}^{-2l(j-p)s_1(\lambda_*)}P_j(\lambda_*,l)z^j=0. \tag{6.5}$$

Due to (5.3) the coefficients of this equation are bounded uniformly in $l$. By Rouche theorem it implies that all the roots of (6.5) are bounded uniformly in $l$. This fact leads us to (1.9). □

In what follows the roots $\tau_i$ are supposed to be ordered in accordance with (1.10). We denote $\mu_i(l):=\lambda_i(l)-\lambda_*$, $i=1,\ldots,p$.

*Proof of Theorem 1.4.* The formulas (1.9), (1.11) were established in Lemma 6.3. Let us prove (1.8). Namely, let us prove that for each $l$ large enough the roots of (1.7) can be ordered so that

$$\tau_i(l)=\mu_i(l)\left(1+\mathcal{O}\left(l^{\frac{2}{p}}\mathrm{e}^{-\frac{4l}{p}s_1(\lambda_*)}\right)\right),\quad l\to+\infty. \tag{6.6}$$

Assume that this not true on a sequence $\lambda_s\to+\infty$. We introduce an equivalence relation $\sim$ on $\{\mu_i(l_s)\}_{i=1,\ldots,p}$ saying that $\mu_i\sim\mu_j$, if

$$\mu_i(l_s)=\mu_j(l_s)\left(1+\mathcal{O}\left(l_s^{\frac{2}{p}}\mathrm{e}^{-\frac{4l_s}{p}s_1(\lambda_*)}\right)\right),\quad l_s\to+\infty.$$

This relation divides all $\mu_i(l_s)$ into disjoint groups,

$$\{\lambda_1(l),\ldots,\lambda_p(l)\}=\bigcup_{i=1}^{q}\{\lambda_{m_i}(l),\ldots,\lambda_{m_{i+1}-1}(l)\},$$

where $1 = m_1 < m_2 < \ldots < m_{q+1} = p+1$, $\lambda_k \sim \lambda_t$, $k,t = m_i, \ldots, m_{i+1}-1$, $i = 1,\ldots,q$, and $\lambda_k \not\sim \lambda_t$, if $m_i \leqslant k \leqslant m_{i+1}-1$, $m_j \leqslant t \leqslant m_{j+1}-1$, $i \neq j$. Extracting if needed a subsequence from $\{l_s\}$, we assume that $m_i$ and $q$ are independent of $\{l_s\}$. Given $k \in \{1,\ldots,p\}$, there exists $i$ such that $m_i \leqslant k \leqslant m_{i+1}-1$. For the sake of brevity we denote $m := m_i$, $\widetilde{m} := m_{i+1}$, $\widehat{m} := \widetilde{m} - m$. To prove (1.8) it is sufficient to show that $\widehat{m}$ roots $\tau_i$, $i = m,\ldots,\widetilde{m}-1$, counting multiplicity of $\det\big(\tau\mathrm{E} - \mathrm{A}(\lambda_*, l)\big)$ satisfy (6.6).

Since

$$\det\big(\tau\mathrm{E} - \mathrm{A}(\lambda_*, l)\big) = \det\big(\mathrm{S}^{-1}(l)(\tau\mathrm{E} - \mathrm{A}(\lambda_*, l))\mathrm{S}(l)\big) = \det\big(\tau\mathrm{E} - \mathrm{R}(\lambda_*, l)\big),$$

due to Lemma 6.2 the equation for $\tau_i$ can be rewritten as

$$\begin{vmatrix} \tau - \mu_1(1-r_{11}) & -\mu_2 r_{12} & \ldots & -\mu_p r_{1p} \\ -\mu_1 r_{21} & \tau - \mu_2(1-r_{22}) & \ldots & -\mu_p r_{2p} \\ \vdots & \vdots & & \vdots \\ -\mu_1 r_{p1} & -\mu_2 r_{p2} & \ldots & \tau - \mu_p(1-r_{pp}) \end{vmatrix} = 0, \tag{6.7}$$

where $\mu_j = \mu_j(l)$, $r_{jk} = r_{jk}(\lambda_*, l)$. Assume first that $\mu_k(l_s) = 0$. In view of (1.6) it implies that $m = m_1$, $\widetilde{m} = m_2$, and $\mu_j(l_s) = 0$, $j = m,\ldots,\widetilde{m}-1$. In this case the equation (6.7) becomes

$$\begin{vmatrix} \tau & \ldots & 0 & -\mu_{\widetilde{m}} r_{1\widetilde{m}} & \ldots & -\mu_p r_{1p} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \ldots & \tau & -\mu_{\widetilde{m}} r_{\widetilde{m}-1\widetilde{m}} & \ldots & -\mu_p r_{\widetilde{m}-1p} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \ldots & 0 & -\mu_{\widetilde{m}} r_{p\widetilde{m}} & \ldots & \tau - \mu_p(1-r_{pp}) \end{vmatrix} = 0,$$

and it implies that zero is the root of $\det\big(\tau\mathrm{E} - \mathrm{A}(\lambda_*, l)\big)$ of multiplicity at least $\widehat{m}$. In this case the identities (6.6) are obviously valid.

Assume now that $\mu_k(l_s) \neq 0$. We seek the needed roots as $\tau = \mu_k(l_s)(1+z)$. We substitute this identity into (6.7) and divide then first $(\widetilde{m}-1)$ columns by $-\mu_k(l_s)$, while the other columns are divided by the functions $-\mu_j(l_s)$ corresponding to them. This procedure leads us to the equation

$$\begin{vmatrix} z_1 - z & \ldots & \frac{\mu_{\widetilde{m}-1}}{\mu_k} r_{1\widetilde{m}-1} & r_{1\widetilde{m}} & \ldots & r_{1p} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\mu_1}{\mu_k} r_{\widetilde{m}-11} & \ldots & z_{\widetilde{m}-1} - z & r_{\widetilde{m}-1\widetilde{m}} & \ldots & r_{\widetilde{m}-1p} \\ \frac{\mu_1}{\mu_k} r_{\widetilde{m}1} & \ldots & \frac{\mu_{\widetilde{m}-1}}{\mu_k} r_{\widetilde{m}\widetilde{m}-1} & z_{\widetilde{m}} - \frac{\mu_k}{\mu_{\widetilde{m}}} z & \ldots & r_{\widetilde{m}p} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\mu_1}{\mu_k} r_{p1} & \ldots & \frac{\mu_{\widetilde{m}-1}}{\mu_k} r_{p\widetilde{m}-1} & r_{p\widetilde{m}} & \ldots & z_p - \frac{\mu_k}{\mu_p} z \end{vmatrix} = 0,$$

$$
\begin{aligned}
z_i = z_i(l) &:= \frac{\mu_i(l)}{\mu_k(l)}(1 - r_{ii}(\lambda_*, l)) - 1, \quad i = 1, \ldots, \widetilde{m} - 1, \\
z_i = z_i(l) &:= 1 - \frac{\mu_k(l)}{\mu_i(l)} - r_{ii}(\lambda_*, l), \qquad i = \widetilde{m}, \ldots, p,
\end{aligned}
$$

and the arguments of all the functions are $l = l_s$, $\lambda = \lambda_*$. Due to (1.6) all the fractions in this determinant are bounded uniformly in $l_s$. Using this fact and (6.1), we calculate this determinant and write the multiplication of the diagonal separately,

$$
\begin{aligned}
&F_1(z, l_s)F_2(z, l_s)F_3(z, l_s) - F_4(z, l_s) = 0, \qquad (6.8) \\
&F_1(z, l_s) := \prod_{i=1}^{m-1} (z_i(l_s) - z)\,, \quad F_2(z, l_s) := \prod_{i=m}^{\widetilde{m}-1} (z_i(l_s) - z)\,, \\
&F_3(z, l_s) := \prod_{i=\widetilde{m}}^{p} \left( z_i(l_s) - \frac{\mu_i(l_s)}{\mu_k(l_s)} z \right), \quad F_4(z, l_s) := \sum_{i=0}^{p-2} z^i c_i(l_s)
\end{aligned}
$$

where the coefficients $c_i$ obey the estimate

$$
c_i(l_s) = \mathcal{O}\Big(l_s^2 \mathrm{e}^{-4 l_s s_1(\lambda_*)}\Big), \quad l_s \to +\infty. \tag{6.9}
$$

As it follows from the definition of the equivalence relation,

$$
\begin{aligned}
&\frac{|\mu_i - \mu_k|}{|\mu_k|} \leqslant C_0\zeta, \quad l_s \to +\infty, \quad i = m, \ldots, \widetilde{m} - 1, \quad \zeta = \zeta(l_s) := l_s^2 \mathrm{e}^{-4 l_s s_1(\lambda_*)}, \\
&\frac{|\mu_i - \mu_k|}{|\mu_k|} \geqslant \theta\zeta, \quad i \notin \{m, \ldots, \widetilde{m} - 1\}, \quad \theta = \theta(l_s) \to +\infty, \quad l_s \to +\infty,
\end{aligned}
$$

where the constant $C_0$ is independent of $l_s$. These estimates and (6.1), (1.6) imply

$$
|F_1(z, l)| \geqslant C(\theta\zeta)^{m-1}, \quad |F_3(z, l)| \geqslant C(\theta\zeta)^{p-\widetilde{m}+1}, \quad |z| \leqslant 2C_0\zeta. \tag{6.10}
$$

The zeroes $z_i(l_s)$ of $F_2(z, l_s)$ satisfy $|z_i(l_s)| \leqslant C_0\zeta$, and hence $|F_2(z, l_s)| \geqslant C_0^{\widehat{m}}\zeta^{\widehat{m}}$ as $|z| = C_0\zeta$. Employing this estimate, (6.10) and rewriting (6.8) as

$$
F_2(z, l_s) - \frac{F_4(z, l_s)}{F_1(z, l_s)F_3(z, l_s)} = 0,
$$

by Rouche theorem we conclude that the last equation has exactly $\widehat{m}$ roots counting multiplicity in the disk $\{|z| < 2C_0\zeta\}$. We denote these roots as $z^{(j)}(l_s)$, $j = m, \ldots, \widetilde{m} - 1$. It follows from (6.8), (6.9), (6.10) that

$$
\begin{aligned}
\left( \min_{i=m,\ldots,\widetilde{m}-1} |z^{(j)} - z_i| \right)^{\widehat{m}} &\leqslant |F_2(z^{(j)}, l_s)| = \left| \frac{F_4(z^{(j)}, l_s)}{F_1(z^{(j)}, l_s)F_3(z^{(j)}, l_s)} \right| \\
&\leqslant C \frac{\zeta^{\widehat{m}}(l_s)}{\theta^{p-\widehat{m}}(l_s)} \leqslant C\zeta^{\widehat{m}}(l_s),
\end{aligned}
$$

where $z^{(j)} = z^{(j)}(l_s)$, and the constant $C$ is independent of $l_s$. Hence, for each $j$ there exists index $i$, depending on $l_s$, such that

$$z^{(j)} = z_i + \mathcal{O}(\zeta), \quad l_s \to +\infty.$$

This identity, the definition of the equivalence relation and (6.1) imply

$$\tau^{(j)} = \mu_k(1 + z^{(j)}) = \mu_j(1 - r_{jj}) + \mathcal{O}(\mu_k\zeta) = \mu_j + \mathcal{O}(\mu_j\zeta),$$

which yields (6.6). $\square$

In the proof of Theorem 1.5 we will employ the following lemma.

**Lemma 6.4.** *For $\lambda$ close to $\lambda_*$ and $\boldsymbol{h} \in \mathbb{C}^p$ the representation*

$$\mathrm{B}^{-1}(\lambda, l)\boldsymbol{h} = \sum_{i=1}^{p} \frac{\mathcal{T}_{10}^{(i)}(\lambda, l)\boldsymbol{h}}{\lambda - \lambda_i(l)}\boldsymbol{k}_i$$

*holds true. Here $\mathcal{T}_{10}^{(i)}(\lambda, l) : \mathbb{C}^p \to \mathbb{C}$ are functionals bounded uniformly in $\lambda$ and $l$.*

*Proof.* Lemma 5.2 implies that for $\lambda$ close to $\lambda_*$ the identity (5.14) holds true for any $\boldsymbol{h} \in \mathbb{C}^p$, where $l_s$ should be replaced by $l$, $q = p$ and $\lambda^{(i)} = \lambda_i$. We introduce the vectors

$$\boldsymbol{k}_i^{\perp}(l) := \frac{\widetilde{\boldsymbol{k}}_i(l)}{\|\widetilde{\boldsymbol{k}}_i(l)\|}, \quad \widetilde{\boldsymbol{k}}_i := \boldsymbol{k}_i - \sum_{\substack{j=1\\ j\neq i}}^{p} (\boldsymbol{k}_i, \boldsymbol{k}_j)_{\mathbb{C}^p}\boldsymbol{k}_j.$$

The relations (1.13) implies that the vectors $\boldsymbol{k}_i^{\perp}$ satisfy these relation as well. Moreover, the vectors $\boldsymbol{k}_i^{\perp}$ form the orthogonal basis for the basis $\{\boldsymbol{k}_i\}$. Bearing this fact in mind, we multiply the relation (5.15) by $\boldsymbol{k}_j^{\perp}$ with $l_s$ replaced by $l$, $q = p$ and obtain

$$\frac{1}{2\pi\mathrm{i}} \left( \int\limits_{|\lambda-\lambda_*|=\delta} \mathrm{B}^{-1}(\lambda, l)\boldsymbol{h}\, \mathrm{d}\lambda, \boldsymbol{k}_j^{\perp} \right)_{\mathbb{C}^p} = \mathcal{T}_8^{(j)}(l)\boldsymbol{h}.$$

Due to (5.16) we conclude that the functionals $\mathcal{T}_8^{(j)}$ are bounded uniformly in $l$.

Let us prove that the matrix $\mathcal{T}_9(\lambda, l)$ is bounded uniformly in $\lambda$ and $l$. Due to (5.16) and the convergences $\lambda_i(\lambda) \to \lambda_*$ we have

$$\|\mathcal{T}_9(\lambda, l)\boldsymbol{h}\|_{\mathbb{C}^p} = \left\| \mathrm{B}^{-1}(\lambda, l)\boldsymbol{h} - \sum_{i=1}^{p} \frac{\mathcal{T}_8^{(i)}(l)\boldsymbol{h}}{\lambda - \lambda_i(l)}\boldsymbol{k}_i(l) \right\|_{\mathbb{C}^p} \leqslant C\|\boldsymbol{h}\|_{\mathbb{C}^p},$$

as $|\lambda - \lambda_*| = \delta$, if $l$ is large enough. The constant $C$ here is independent of $\boldsymbol{h}$ and $\lambda$ such that $|\lambda - \lambda_*| = \delta$. The matrix $\mathcal{T}_9$ being holomorphic w.r.t. $\lambda$, by the maximum principle for holomorphic functions the estimate holds true for $|\lambda - \lambda_*| < \delta$, too.

Thus, the matrix $\mathcal{T}_9$ is bounded uniformly in $\lambda$ and $l$. We can expand $\mathcal{T}_9\boldsymbol{h}$ in terms of the basis $\{\boldsymbol{k}_i\}$,

$$\mathcal{T}_9(\lambda,l)\boldsymbol{h}=\sum_{i=1}^{p}\boldsymbol{k}_i\mathcal{T}_{11}^{(i)}(\lambda,l)\boldsymbol{h},\quad \left(\mathcal{T}_{11}^{(1)}\boldsymbol{h}\ldots\mathcal{T}_{11}^{(p)}\boldsymbol{h}\right)^t=\mathrm{S}^{-1}(l)\mathcal{T}_9(\lambda,l)\boldsymbol{h},\tag{6.11}$$

where the functionals $\mathcal{T}_{11}^{(i)}:\mathbb{C}^p\to\mathbb{C}$ are bounded uniformly in $\lambda$ and $l$. Substituting (6.11) into (5.14) with $l_s$ replaced by $l$, $q=p$, $\lambda^{(i)}=\lambda_i$, we arrive at the desired representation, where $\mathcal{T}_{10}^{(i)}(\lambda,l)=\mathcal{T}_8^{(i)}(l)+(\lambda-\lambda_i(l))\mathcal{T}_{11}^{(i)}(\lambda,l)$. $\square$

*Proof of Theorem 1.5.* Since the matrix $\mathrm{A}_0(l)$ satisfies the condition (A), it has $p$ eigenvalues $\tau_i^{(0)}$, $i=1,\ldots,p$, counting multiplicity. By $\boldsymbol{h}_i=\boldsymbol{h}_i(l)$, $i=1,\ldots,p$, we denote the associated eigenvectors normalized in $\mathbb{C}^p$. Completely by analogy with the proof of (1.9) in Lemma 6.3 one can establish (1.16). It is easy to check that the vectors $\boldsymbol{h}_i$ satisfy the identities

$$\mathrm{B}(\lambda_*+\tau_i^{(0)},l)\boldsymbol{h}_i(l)=-\mathrm{A}_1(l)\boldsymbol{h}_i(l)-\tau_i^{(0)}\widetilde{\mathrm{A}}(\lambda_*,\lambda_*+\tau_i^{(0)},l)\boldsymbol{h}_i(l):=\widetilde{\boldsymbol{h}}_i(l),$$

where, we remind, the matrix $\widetilde{\mathrm{A}}$ was introduced in (6.2). Now we employ (5.2) to obtain

$$\|\widetilde{\boldsymbol{h}}_i(l)\|_{\mathbb{C}^p}\leqslant\|\mathrm{A}_1(l)\|+C|\tau_i^{(0)}|l^2\mathrm{e}^{-2l(s_1(\lambda_*)+s_1(\widetilde{\lambda}_i(l)))}\leqslant\|\mathrm{A}_1(l)\|+C|\tau_i^{(0)}|l^2\mathrm{e}^{-4ls_1(\lambda_*)},\tag{6.12}$$

where the constant $C$ is independent of $\lambda$ and $l$. Here we have also used the identities

$$\widetilde{\lambda}_i(l)=\lambda_*+\mathcal{O}(\mathrm{e}^{-2ls_1(\lambda_*)}),\quad \mathrm{e}^{-2ls_1(\widetilde{\lambda}_i(l))}=\mathcal{O}(\mathrm{e}^{-2ls_1(\lambda_*)}),\quad l\to+\infty,$$

which are due to (1.8), (1.9).

Since $\boldsymbol{h}_i(l)=\mathrm{B}^{-1}(\lambda_*+\tau_i^{(0)}(l),l)\widetilde{\boldsymbol{h}}_i(l)$, Lemma 6.4 implies

$$\boldsymbol{h}_j(l)=\sum_{i=1}^{p}\frac{\mathcal{T}_{10}^{(i)}(\lambda_*+\tau_j^{(0)}(l),l)\widetilde{\boldsymbol{h}}_i(l)}{\tau_i^{(0)}(l)+\lambda_*-\lambda_j(l)}\boldsymbol{k}_i(l),\quad j=1,\ldots,p.$$

The fractions in these identities are bounded uniformly in $\lambda$ and $l$ since $\boldsymbol{h}_j$ are normalized and

$$Q_{ji}(l):=\frac{\mathcal{T}_{10}^{(i)}(\lambda_*+\tau_j^{(0)}(l),l)\widetilde{\boldsymbol{h}}_j(l)}{\tau_i^{(0)}(l)+\lambda_*-\lambda_i(l)}=\big(\boldsymbol{h}_j(l),\boldsymbol{k}_i^{\perp}(l)\big)_{\mathbb{C}^p},$$

where, we remind, the vectors $\boldsymbol{k}_i^{\perp}$ were introduced in the proof of Lemma 6.4. We are going to prove that the roots $\tau_j^{(0)}$ can be ordered so that the formulas (1.15) hold true. The matrices

$$\mathrm{Q}:=\begin{pmatrix}Q_{11}&\ldots&Q_{1p}\\ \vdots&&\vdots\\ Q_{p1}&\ldots&Q_{pp}\end{pmatrix},\quad \mathrm{K}=(\boldsymbol{k}_1^{\perp}\ldots\boldsymbol{k}_p^{\perp}),\quad \mathrm{H}=(\boldsymbol{h}_1\ldots\boldsymbol{h}_p)^t,$$

satisfy the identity $\mathrm{H}\overline{\mathrm{K}} = \mathrm{Q}$. The basis $\{\boldsymbol{k}_i^{\perp}\}$ being orthogonal to $\{\boldsymbol{k}_i\}$, we conclude that $\mathrm{K}^t = \mathrm{S}^{-1}$. Now Lemma 6.1 and the condition (A) for $\mathrm{A}_0(l)$ imply the uniform in $l$ estimate $|\det Q| \geqslant C_0 > 0$. Hence, there exists a permutation

$$\eta = \eta(l), \quad \eta = \begin{pmatrix} 1 & \dots & p \\ \eta_1 & \dots & \eta_p \end{pmatrix}$$

such that

$$\prod_{i=1}^{p} |Q_{i\eta_i}| \geqslant \frac{C_0}{p!}\,. \tag{6.13}$$

This fact is proved easily by the contradiction employing the definition of the determinant. The normalization of $\boldsymbol{h}_i$ and the identities (1.13) for $\boldsymbol{k}_i^{\perp}$ imply that $|Q_{i\eta_i}| \leqslant \|\boldsymbol{h}_i\|_{\mathbb{C}^p} \|\boldsymbol{k}_{\eta_i}^{\perp}\|_{\mathbb{C}^p} = 1$. Hence, by (6.13) we obtain

$$\frac{C_0}{|Q_{j\eta_j}|p!} \leqslant \prod_{\substack{i=1\\ i\neq j}}^{p} |Q_{i\eta_i}| \leqslant 1, \quad \frac{C_0}{p!} \leqslant |Q_{j\eta_j}|.$$

Substituting the definition of $Q_{i\eta_i}$ in this estimate, we have

$$|\lambda_i - \lambda_* - \tau_i^{(0)}| \leqslant C \left| \mathcal{T}_{10}^{(i)}(\lambda_* + \tau_{\eta_i}^{(0)}, l)\widetilde{\boldsymbol{h}}_i \right| \leqslant C\big(\|\mathrm{A}_1\| + |\tau_{\eta_i}^{(0)}| l^2 \mathrm{e}^{-4ls_1(\lambda_*)}\big).$$

Here we have also used the boundedness of $\mathcal{T}_{10}^{(i)}$ (see Lemma 6.4) and (6.12). Rearranging the roots $\tau_i^{(0)}$ as $\tau_i^{(0)} := \tau_{\eta_i}^{(0)}$, we complete the proof. $\square$

*Proof of Theorem 1.8.* Let us prove first that the matrix $\mathrm{A}(\lambda_*, l)$ obeys the representation (1.14), where all the elements of $\mathrm{A}_0$ are zero except ones standing on the intersection of the first row and of $(p_- + 1)$-th column and $(p_- + 1)$-th column and the first row, and these elements are given by

$$A_{0,1p_-+1}(l) = \overline{A}_{0,p_-+11}(l) = 2s_1(\lambda_*)\overline{\beta}_-\beta_+ \mathrm{e}^{-2ls_1(\lambda_*)}.$$

Also we are going to prove that the corresponding matrix $\mathrm{A}_1$ satisfies the estimate

$$\|\mathrm{A}_1(l)\| = \mathcal{O}(l\mathrm{e}^{-4l\sqrt{\nu_1 - \lambda_*}}), \quad l \to +\infty. \tag{6.14}$$

The definition of $\Phi_i$, the formulas (3.11) for $\varphi_j^{\pm}$, (4.16), (4.7) imply that

$$\boldsymbol{\Phi}_i(\cdot, \lambda_*, l) = \boldsymbol{\phi}_i(\cdot, l) + \mathcal{O}\big(l\mathrm{e}^{-4ls_1(\lambda_*)}\big), \quad l \to +\infty, \tag{6.15}$$

in $L_2(\Omega_-) \oplus L_2(\Omega_+)$-norm. The identity (6.15) and the definition of $A_{ij}$ and $\mathcal{T}_1^{(i)}$ yield that for $i, j = 1, \dots, p_-$

$$A_{ij}(\lambda_*, l) = \mathcal{T}_1^{(i)}\Phi_j(\cdot, \lambda_*, l) = \mathcal{T}_1^{(i)}\phi_j(\cdot, l) + \mathcal{O}\big(l\mathrm{e}^{-4ls_1(\lambda_*)}\big) = \mathcal{O}\big(l\mathrm{e}^{-4ls_1(\lambda_*)}\big), \tag{6.16}$$

since $\mathcal{T}_1^{(i)}\phi_j(\cdot,l)=0$, $i,j=1,\ldots,p_-$. In the same way one can check easily the same identity for $i,j=p_-+1,\ldots,p$. Taking into account the definition (3.11) of $\varphi_1^-$, and (1.17) by direct calculations we obtain that

$$\begin{aligned}
&A_{1p_-+1}(\lambda_*,l)=\big(\psi_1^+,\varphi_1^-(\cdot,l)\big)_{L_2(\Omega_+)}=\big(\psi_1^+,\mathcal{L}_+\mathcal{S}(2l)\psi_1^-\big)_{L_2(\Omega_+)}=\\
&=\overline{\beta}_-\mathrm{e}^{-2ls_1(\lambda_*)}\big(\psi_1^+,\mathcal{L}_+\mathrm{e}^{-s_1(\lambda_*)x_1}\phi_1\big)_{L_2(\Omega_+)}+\mathcal{O}\big(\mathrm{e}^{-2(s_1l(\lambda_*)+s_2(\lambda_*))}\big),\quad l\to+\infty.
\end{aligned}$$

Now we use the definition of $\mathcal{L}_+$, (1.17), and integrate by parts,

$$\begin{aligned}
&\big(\psi_1^+,\mathcal{L}_+\mathrm{e}^{-s_1(\lambda_*)x_1}\phi_1\big)_{L_2(\Omega_+)}=\big(\psi_1^+,(-\Delta-\lambda_*+\mathcal{L}_+)\mathrm{e}^{-s_1(\lambda_*)x_1}\phi_1\big)_{L_2(\Pi)}\\
&=\lim_{x_1\to-\infty}\left(\left(\mathrm{e}^{-s_1(\lambda_*)x_1}\overline{\phi}_1,\frac{\partial\psi_1^+}{\partial x_1}\right)_{L_2(\omega)}-\left(\psi_1^+,\frac{\partial}{\partial x_1}\mathrm{e}^{-s_1(\lambda_*)x_1}\overline{\phi}_1\right)_{L_2(\omega)}\right)\\
&=2s_1(\lambda_*)\beta_+,
\end{aligned}\tag{6.17}$$

which together with previous formula implies

$$A_{1p_-+1}(\lambda_*,l)=2\overline{\beta}_-\beta_+s_1(\lambda_*)\mathrm{e}^{-2ls_1(\lambda_*)}+\mathcal{O}\big(\mathrm{e}^{-2l(s_1(\lambda_*)+s_2(\lambda_*))}\big),\quad l\to+\infty.\tag{6.18}$$

In the same way one can show that

$$\begin{aligned}
&A_{1j}(\lambda_*,l)=\mathcal{O}\big(\mathrm{e}^{-2l(s_1(\lambda_*)+s_2(\lambda_*))}\big),\quad j=p_-+2,\ldots,p,\\
&A_{ij}(\lambda_*,l)=\mathcal{O}\big(\mathrm{e}^{-4ls_2(\lambda_*)}\big),\quad i=2,\ldots,p_-,\quad j=p_-+1,\ldots,p,\\
&A_{p_-+11}(\lambda_*,l)=2\beta_-\overline{\beta}_+s_1(\lambda_*)\mathrm{e}^{-2ls_1(\lambda_*)}+\mathcal{O}\big(\mathrm{e}^{-2l(s_1(\lambda_*)+s_2(\lambda_*))}\big),\\
&A_{p_-+1j}(\lambda_*,l)=\mathcal{O}\big(\mathrm{e}^{-2l(s_1(\lambda_*)+s_2(\lambda_*))}\big),\quad j=2,\ldots,p_-,\\
&A_{ij}(\lambda_*,l)=\mathcal{O}\big(\mathrm{e}^{-4ls_2(\lambda_*)}\big),\quad i=p_-+2,\ldots,p,\quad j=1,\ldots,p_-,
\end{aligned}$$

as $l\to+\infty$. The formulas obtained and (6.16), (6.18) lead us to the representation (1.14), where the matrix $\mathrm{A}_0$ is defined as described, while the matrix $\mathrm{A}_1$ satisfies (6.14). The matrix $\mathrm{A}_0$ being hermitian, it satisfies the condition (A). This fact can be proved completely by analogy with Lemma 6.1.

Let us calculate the roots of $\det(\tau\mathrm{E}-\mathrm{A}_0)$. For the sake of brevity within the proof we denote $c:=-2\overline{\beta}_-\beta_+s_1(\lambda_*)\mathrm{e}^{-2ls_1(\lambda_*)}$. Expanding the determinant $\det(\tau\mathrm{E}-\mathrm{A}_0)$ w.r.t. the first column, one can make sure that

$$\det(\tau\mathrm{E}-\mathrm{A}_0)=\tau^{p-2}(\tau^2-|c|^2).$$

Thus, $\tau_0^{(1)}(l)=\ldots=\tau_0^{(p-2)}(l)=0$ is a root of multiplicity $(p-2)$, and $\tau_0^{(p-1)}(l)=-2|c|$, $\tau_0^{(p)}(l)=2|c|$. Applying Theorem 1.5, we arrive at (1.21). □

*Proof of Theorem 1.6.* As in the proof of Theorem 1.8, we begin with the proving (1.14). Namely, we are going to show that

$$\mathrm{A}_0(l)=\operatorname{diag}\{A_{11}(\lambda_*,l),0,\ldots,0\},\quad\|\mathrm{A}_1(l)\|=\mathcal{O}\big(\mathrm{e}^{-2l(s_1(\lambda_*)+s_2(\lambda_*))}\big).\tag{6.19}$$

The definition of A implies

$$A_{ij}(\lambda_*,l) = -\big(\mathcal{T}_6^-(\lambda_*,l)(\mathrm{I}-\mathcal{T}_7^+(\lambda_*,l)\mathcal{T}_6^-(\lambda_*,l))^{-1}\varphi_i^-(\cdot,l),\psi_j^-\big)_{L_2(\Pi)}. \tag{6.20}$$

Employing the definition (3.9) of $\mathcal{T}_6^-$, for any $f\in L_2(\Pi,\Omega_+)$ we check that

$$\begin{aligned}
&\big(\mathcal{T}_6^-(\lambda_*,l)f,\psi_j^-\big)_{L_2(\Pi)} = \big(\mathcal{L}_-\mathcal{S}(-2l)(\mathcal{H}_+-\lambda_*)^{-1}f,\psi_j^-\big)_{L_2(\Pi)}\\
&=\big(\mathcal{S}(-2l)(\mathcal{H}_+-\lambda_*)^{-1}f,\mathcal{L}_-\psi_j^-\big)_{L_2(\Pi)} = \big((\mathcal{H}_+-\lambda_*)^{-1}f,\mathcal{S}(2l)(\Delta+\lambda_*)\psi_j^-\big)_{L_2(\Pi)}\\
&=-\big((\mathcal{H}_+-\lambda_*)^{-1}f,(\mathcal{H}_+-\lambda_*-\mathcal{L}_+)\mathcal{S}(2l)\psi_j^-\big)_{L_2(\Pi)}\\
&=\big(f,(\mathcal{H}_+-\lambda_*)^{-1}\mathcal{L}_+\mathcal{S}(2l)\psi_j^- - \mathcal{S}(2l)\psi_j^-\big)_{L_2(\Pi)}.
\end{aligned}$$

Using this identity, (6.20), (3.10), (3.12), and Lemma 1.1, we obtain that for $(i,j)\neq(1,1)$ the functions $A_{ij}$ satisfy the relation

$$\begin{aligned}
|A_{ij}| &\leqslant \|(\mathrm{I}-\mathcal{T}_7^+\mathcal{T}_6^-)^{-1}\varphi_i^-\|_{L_2(\Omega_+)}\|(\mathcal{H}_+-\lambda_*)^{-1}\mathcal{L}_+\mathcal{S}(2l)\psi_j^- - \mathcal{S}(2l)\psi_j^-\|_{L_2(\Omega_+)} =\\
&=\mathcal{O}\big(\mathrm{e}^{-2l(s_1(\lambda_*)+s_2(\lambda_*))}\big),\quad l\to+\infty,
\end{aligned}$$

where in the arguments $\lambda=\lambda_*$, and the operator $(\mathcal{H}_+-\lambda_*)^{-1}$ is bounded since $p_+=0$. The identities (6.19) therefore hold true.

We apply now Theorem 1.5 and infer that the formulas (1.18) are valid for the eigenvalues $\lambda_i$, $i=1,\ldots,p-1$, while the eigenvalue $\lambda_p$ satisfies

$$\lambda_p(l) = \lambda_* + A_{11}(\lambda_*,l)\big(1+\mathcal{O}(l^2\mathrm{e}^{-2ls_1(\lambda_*)})\big)+\mathcal{O}\big(\mathrm{e}^{-2l(s_1(\lambda_*)+s_2(\lambda_*))}\big),\quad l\to+\infty. \tag{6.21}$$

Now it is sufficient to find out the asymptotic behaviour of $A_{11}(\lambda_*,l)$.

The formula (6.20) for $A_{11}$, Lemma 1.1, and (3.10), (3.12) yield

$$\begin{aligned}
&A_{11}(\lambda_*,l) = -\big(\mathcal{T}_6^-(\lambda_*,l)\varphi_1^-(\cdot,l),\psi_1^-\big)_{L_2(\Pi)}+\mathcal{O}\big(l\mathrm{e}^{-8ls_1(\lambda_*)}\big)\\
&=\beta_-\mathrm{e}^{-2ls_1(\lambda_*)}\big(\mathcal{T}_6^-(\lambda_*,l)\mathcal{L}_+\mathrm{e}^{-s_1(\lambda_*)x_1},\psi_1^-\big)_{L_2(\Pi)}+\mathcal{O}\big(\mathrm{e}^{-2l(s_1(\lambda_*)+s_2(\lambda_*))}+l\mathrm{e}^{-8ls_1(\lambda_*)}\big),\\
&\big(\mathcal{T}_6^-(\lambda_*,l)\mathcal{L}_+\mathrm{e}^{-s_1(\lambda_*)x_1},\psi_1^-\big)_{L_2(\Pi)} = \big(\mathcal{L}_-\mathcal{S}(-2l)(\mathcal{H}_+-\lambda_*)^{-1}\mathcal{L}_+\mathrm{e}^{-s_1(\lambda_*)x_1},\psi_1^-\big)_{L_2(\Omega_-)}\\
&=\big(\mathcal{S}(-2l)(\mathcal{H}_+-\lambda_*)^{-1}\mathcal{L}_+\mathrm{e}^{-s_1(\lambda_*)x_1},\mathcal{L}_-\psi_1^-\big)_{L_2(\Omega_-)} = \big(\mathcal{S}(-2l)U_+,\mathcal{L}_-\psi_1^-\big)_{L_2(\Omega_-)}.
\end{aligned}$$

Lemma 3.2 implies that the function $U$ satisfies (1.19) that determines the constant $\widetilde{\beta}_-$ uniquely. Employing (1.19), we continue our calculations,

$$\big(\mathcal{S}(-2l)U_+,\mathcal{L}_-\psi_1^-\big)_{L_2(\Omega_-)} = \widetilde{\beta}_-\mathrm{e}^{-2ls_1(\lambda_*)}\big(\mathrm{e}^{s_1(\lambda_*)x_1}\phi_1(x'),\mathcal{L}_-\psi_1^-\big)_{L_2(\Pi)}+\mathcal{O}(\mathrm{e}^{-2ls_2(\lambda_*)}).$$

Integrating by parts in the same way as in (6.17), we obtain

$$\big(\mathrm{e}^{s_1(\lambda_*)x_1}\phi_1(x'),\mathcal{L}_-\psi_1^-\big)_{L_2(\Pi)} = \big(\mathcal{L}_-\mathrm{e}^{s_1(\lambda_*)x_1}\phi_1(x'),\psi_1^-\big)_{L_2(\Pi)} = -2s_1(\lambda_*)\overline{\beta}_-.$$

Therefore,

$$A_{11}(\lambda_*,l) = 2s_1(\lambda_*)|\beta_-|^2\widetilde{\beta}_-\mathrm{e}^{-4ls_1(\lambda_*)}+\mathcal{O}\big(\mathrm{e}^{-2l(s_1(\lambda_*)+s_2(\lambda_*))}+l\mathrm{e}^{-8ls_1(\lambda_*)}\big).$$

Substituting this identity into (6.21), we arrive at the required formula for $\lambda_p$. $\square$

The proof of Theorem 1.7 is completely analogous to that of Theorem 1.6.

# 7 Examples

In this section we provide some examples of the operators $\mathcal{L}_\pm$. In what follows we will often omit the index "$\pm$" in the notation $\mathcal{L}_\pm$, $\mathcal{H}_\pm$, $\Omega_\pm$, $a_\pm$, writing $\mathcal{L}$, $\mathcal{H}$, $\Omega$, $a$ instead.

**1. Potential.** The simplest example is the multiplication operator $\mathcal{L} = V$, where $V = V(x) \in C(\overline{\Pi})$ is a real-valued compactly supported function. Although this example is classical one for the problems in the whole space, to our knowledge, the double-well problem in waveguide has not been considered yet.

**2. Second order differential operator.** This is a generalization of the previous example. We introduce the operator $\mathcal{L}$ as

$$\mathcal{L} = \sum_{i,j=1}^{n} b_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + b_0, \tag{7.1}$$

where the complex-valued functions $b_{ij} = b_{ij}(x)$ are piecewise continuously differentiable in $\overline{\Pi}$, $b_i = b_i(x)$ are complex-valued functions piecewise continuous in $\overline{\Pi}$. These functions are assumed to be compactly supported. The only restriction to the functions are the conditions (1.1), (1.2); the self-adjointness of $\mathcal{H}$ and $\mathcal{H}_l$ is implied by these conditions. One of the possible way to choose the functions in (7.1) is as follows

$$\mathcal{L} = \operatorname{div} \mathrm{G}\nabla + \mathrm{i} \sum_{i=1}^{n} \left( b_i \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i} b_i \right) + b_0, \tag{7.2}$$

where $\mathrm{G} = \mathrm{G}(x)$ is $n \times n$ hermitian matrix with piecewise continuously differentiable coefficients, $b_i = b_i(x)$ are real-valued piecewise continuously differentiable functions, $b_0 = b_0(x)$ is a real-valued piecewise continuous function. The matrix G and the functions $b_i$ are assumed to be compactly supported and

$$(\mathrm{G}(x)y, y)_{\mathbb{C}^n} \geqslant -c_0 \|y\|^2_{\mathbb{C}^n}, \qquad x \in \overline{\Pi}, \quad y \in \mathbb{C}^n.$$

The constant $c_0$ is independent of $x$, $y$ and satisfies (1.3). The matrix G is not necessarily non-zero. In the case $\mathrm{G} = 0$ one has an example of a first order differential operator.

**3. Magnetic Schrödinger operator.** This is the example with a compactly supported magnetic field. The operator $\mathcal{L}$ is given by (7.2), where $\mathrm{G} = 0$. The coefficients $b_j$ form a magnetic real-valued vector-potential $\boldsymbol{b} = (b_1, \ldots, b_n) \in C^1(\overline{\Pi})$, and $b_0 = \|\boldsymbol{b}\|^2_{\mathbb{C}^n} + V$, where $V = V(x) \in C(\overline{\Pi})$ is a compactly supported real-valued electric potential. The main assumption is the identities

$$\frac{\partial b_j}{\partial x_i} = \frac{\partial b_i}{\partial x_j}, \qquad x \in \overline{\Pi} \setminus \Omega, \quad i, j = 1, \ldots, n. \tag{7.3}$$

To satisfy the conditions required for $\mathcal{L}$, the magnetic vector potential should have a compact support. We are going to show that one can always achieve it by employing the gauge invariance.

The operator $-\Delta^{(D)}+\mathcal{L}$ can be represented as $-\Delta^{(D)}+\mathcal{L}=(\mathrm{i}\nabla+\boldsymbol{b})^2+V$, and for any $\beta=\beta(x)\in C^2(\overline{\Pi})$ the identity

$$\mathrm{e}^{-\mathrm{i}\beta}(\mathrm{i}\nabla+\boldsymbol{b})^2\mathrm{e}^{\mathrm{i}\beta}=(\mathrm{i}\nabla+\boldsymbol{b}-\nabla\beta)^2$$

holds true. In view of (7.3) we conclude that there exist two functions $\beta_\pm=\beta_\pm(x)$ belonging to $C^2(\overline{\Pi\cap\{x:\pm x_1>a\}})$ such that $\nabla\beta_\pm=\boldsymbol{b}$, $\quad x\in\Pi\cap\{x:\pm x_1>a\}$. We introduce now the function $\beta$ as

$$\beta(x)=\begin{cases}\chi(a+x_1+1)\beta_-(x), & x_1\in(-\infty,-a),\quad x'\in\omega,\\ 0, & x_1\in[-a,a],\quad x'\in\omega,\\ \chi(a-x_1+1)\beta_+(x), & x_1\in(a,+\infty),\quad x'\in\omega,\end{cases}$$

where, we remind, the cut-off function $\chi$ was introduced in the proof of Lemma 2.1. Clearly, $\beta\in C^2(\overline{\Pi})$, and $\nabla\beta=\boldsymbol{b}$, $x\in\overline{\Pi}\setminus\{x:|x_1|<a+1\}$. Therefore, the vector $\boldsymbol{b}-\nabla\beta$ has compact support.

If one of the distant perturbations in the operator $\mathcal{H}_l$, say, the right one, is a compactly supported magnetic field, it is sufficient to employ the gauge transformation $\psi(x)\mapsto\mathrm{e}^{\mathrm{i}\beta(x_1-l,x')}\widetilde{\psi}(x)$ to satisfy the conditions for $\mathcal{L}_+$.

**4. Curved and deformed waveguide.** One more interesting example is a geometric perturbation. Quite popular cases are local deformation of the boundary and curving the waveguide (see, for instance, [7], [6], [2], and references therein). Here we consider the case of general geometric perturbation, which includes in particular deformation and curving. Namely, let $\widetilde{x}=\mathcal{G}(x)\in C^2(\overline{\Pi})$ be a diffeomorphism, where $\widetilde{x}=(\widetilde{x}_1,\ldots,\widetilde{x}_n)$, $\mathcal{G}(x)=(\mathcal{G}_1(x),\ldots,\mathcal{G}_n(x))$. We denote $\widetilde{\Pi}:=\mathcal{G}(\Pi)$, $\mathcal{P}:=\mathcal{G}^{-1}$. By $\mathrm{P}=\mathrm{P}(x)$ we indicate the matrix

$$\mathrm{P}:=\begin{pmatrix}\frac{\partial\mathcal{P}_1}{\partial\widetilde{x}_1} & \cdots & \frac{\partial\mathcal{P}_1}{\partial\widetilde{x}_n}\\ \vdots & & \vdots\\ \frac{\partial\mathcal{P}_n}{\partial\widetilde{x}_1} & \cdots & \frac{\partial\mathcal{P}_n}{\partial\widetilde{x}_n}\end{pmatrix},$$

while the symbol $\mathrm{p}=\mathrm{p}(x)$ denotes the corresponding Jacobian, $\mathrm{p}(x):=\det\mathrm{P}(x)$. The function $\mathrm{p}(x)$ is supposed to have no zeroes in $\overline{\Pi}$. The main assumption we make is

$$\mathrm{P}(x)=\mathit{const},\qquad \mathrm{P}^t\mathrm{P}=\mathrm{E},\quad x\in\overline{\Pi}\setminus\Omega. \tag{7.4}$$

It implies that outside $\Omega$ the mapping $\mathcal{P}$ acts as a combination of a shift and a rotating. Hence, the part of $\widetilde{\Pi}$ given by $\mathcal{G}((a,+\infty)\times\omega)$ is also a tubular domain being a direct product of a half-line and $\omega$. The same is true for $\mathcal{G}((-\infty,-a)\times\omega)$. The typical example of the domain $\widetilde{\Pi}$ is given on Figure 1.

Let $-\widetilde{\Delta}^{(D)}$ be the negative Dirichlet Laplacian in $L_2(\widetilde{\Pi})$ with the domain $W^2_{2,0}(\widetilde{\Pi})$. It is easy to check that the operator $\mathcal{U}:L_2(\widetilde{\Pi})\to L_2(\Pi)$ defined as

$$(\mathcal{U}v)(x):=\mathrm{p}^{-1/2}(x)v(\mathcal{P}^{-1}(x)) \tag{7.5}$$

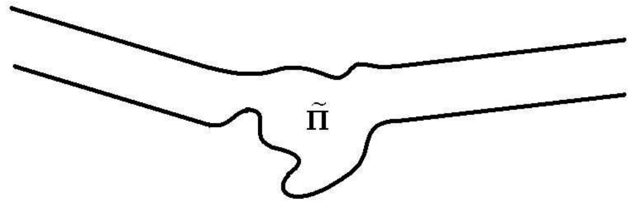

Figure 1: Geometric perturbation

is unitary. The operator $\mathcal{H} := -\mathcal{U}\widetilde{\Delta}^{(D)}\mathcal{U}^{-1}$ has $W_{2,0}^2(\Pi)$ as the domain and is self-adjoint in $L_2(\Pi)$. It can be represented as $\mathcal{H} = -\Delta^{(D)} + \mathcal{L}$, where $\mathcal{L}$ is a second order differential operator

$$\mathcal{L} = -\mathrm{p}^{1/2}\operatorname{div}_x \mathrm{p}^{-1}\mathrm{P}^t\mathrm{P}\nabla_x \mathrm{p}^{1/2} + \Delta_x. \tag{7.6}$$

Indeed, for any $u_1, u_2 \in C_0^\infty(\Pi)$

$$\begin{aligned}(\mathcal{H}u_1, u_2)_{L_2(\Pi)} &= -(\widetilde{\Delta}^{(D)}\mathcal{U}^{-1}u_1, \mathcal{U}^{-1}u_2)_{L_2(\widetilde{\Pi})} = \int\limits_{\widetilde{\Pi}} \big(\nabla_{\widetilde{x}}\mathrm{p}^{1/2}u_1, \nabla_{\widetilde{x}}\mathrm{p}^{1/2}u_2\big)_{\mathbb{C}^n}\, d\widetilde{x} = \\ &= \int\limits_{\Pi} \big(\mathrm{P}\nabla_x\mathrm{p}^{1/2}u_1, \mathrm{p}^{-1}\mathrm{P}\nabla_x\mathrm{p}^{1/2}u_2\big)_{\mathbb{C}^n}\, dx = -\int\limits_{\Pi} \overline{u}_2\mathrm{p}^{1/2}\operatorname{div}_x \mathrm{p}^{-1}\mathrm{P}^t\mathrm{P}\nabla_x\mathrm{p}^{1/2}u_1\, dx.\end{aligned} \tag{7.7}$$

The assumption (7.4) yields that $\mathrm{p} = 1$ holds for $x \in \overline{\Pi} \setminus \Omega$, and therefore the coefficients of the operator $\mathcal{L}$ have the support inside $\overline{\Omega}$.

We are going to check the conditions (1.1), (1.2), (1.3) for the operator $\mathcal{L}$ introduced by (7.6). The symmetricity is obvious, while the estimates follow from (7.6), (7.7),

$$\begin{aligned}(\mathcal{L}u, u)_{L_2(\Omega)} &= \|\mathrm{p}^{-1/2}\mathrm{P}\nabla_x\mathrm{p}^{1/2}u\|^2_{L_2(\Omega)} - \|\nabla u\|^2_{L_2(\Omega)} \geqslant C\|\nabla_x\mathrm{p}^{1/2}u\|^2_{L_2(\Omega)} - \|\nabla u\|^2_{L_2(\Omega)} \\ &= C\left(\|\mathrm{p}^{1/2}\nabla_x u\|^2_{L_2(\Omega)} + \|u\nabla\mathrm{p}^{1/2}\|^2_{L_2(\Omega)} + 2(\mathrm{p}^{1/2}\nabla_x u, u\nabla\mathrm{p}^{1/2})_{L_2(\Omega)}\right) - \|\nabla u\|^2_{L_2(\Omega)} \\ &\geqslant C\left(\frac{1}{2}\|\mathrm{p}^{1/2}\nabla_x u\|^2_{L_2(\Omega)} - \|u\nabla\mathrm{p}^{1/2}\|^2_{L_2(\Omega)}\right) - \|\nabla u\|^2_{L_2(\Omega)} \\ &\geqslant -\left(1 - \frac{C}{2}\right)\|\nabla_x u\|^2_{L_2(\Omega)} - C\|u\|^2_{L_2(\Omega)},\end{aligned}$$

where $C > 0$ is a constant.

If both the operators $\mathcal{L}_\pm$ are the geometric perturbations described by the diffeomorphisms $\mathcal{G}^\pm(x)$, without loss of generality we can assume that $\mathcal{G}^\pm(x) \equiv x$ as $\pm x_1 < -a_\pm$. We introduce now one more diffeomorphism

$$\mathcal{G}_l(x) := \begin{cases}\mathcal{G}^+(x_1 - l, x') + (l, 0, \ldots, 0), & x_1 \geqslant 0, \\ \mathcal{G}^-(x_1 + l, x') - (l, 0, \ldots, 0), & x_1 \leqslant 0,\end{cases}$$

This mapping is well-defined for $l > \max\{a_-, a_+\}$. The domain $\widetilde{\Pi}_l := \mathcal{G}_l(\Pi)$ can be naturally regarded as a waveguide with two distant geometric perturbations. Considering the negative Dirichlet Laplacian in $L_2(\widetilde{\Pi})$, we obtain easily an unitary equivalent operator in $L_2(\Pi)$. The corresponding unitary operator is defined by (7.5) with $\mathcal{P}$ replaced by $\mathcal{G}_l^{-1}$ and the similar replacement for p is required. Clearly, the obtained operator in $L_2(\Pi)$ is the operator $\mathcal{H}_l$ generated by the operators $\mathcal{L}_\pm$ associated with $\mathcal{G}^\pm$.

**5. Delta interaction.** Our next example is the delta interaction supported by a manifold. Let $\Gamma$ be a closed bounded $C^3$-manifold in $\Pi$ of the codimension one and oriented by a normal vector field $\boldsymbol{\nu}$. It is supposed that $\Gamma \cap \partial\Pi = \emptyset$. The manifold $\Gamma$ can consist of several components. By $\xi = (\xi_1, \ldots, \xi_{n-1})$ we denote coordinates on $\Gamma$, while $\widetilde{\varrho}$ will indicate the distance from a point to $\Gamma$ measured in the direction of $\boldsymbol{\nu} = \boldsymbol{\nu}(\xi)$. We assume that $\Gamma$ is so that the coordinates $(\widetilde{\varrho}, \xi)$ are well-defined in a neighbourhood of $\Gamma$. Namely, we suppose that the mapping $(\varrho, \xi) = \mathcal{P}_\Gamma(\widetilde{x})$ is $C^3$-diffeomorphism, where $\widetilde{x} = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ are the coordinates in $\Pi$. Let $b = b(\xi) \in C^3(\Gamma)$ be a real-valued function. The operator in question is the negative Laplacian defined on the functions $v \in W_2^2(\Pi \setminus \Gamma, \partial\Pi) \cap W_{2,0}^1(\Pi)$ satisfying the condition

$$v\big|_{\widetilde{\varrho}=+0} - v\big|_{\widetilde{\varrho}=-0} = 0, \quad \frac{\partial v}{\partial \widetilde{\varrho}}\bigg|_{\widetilde{\varrho}=+0} - \frac{\partial v}{\partial \widetilde{\varrho}}\bigg|_{\widetilde{\varrho}=-0} = bv\big|_{\widetilde{\varrho}=0}. \tag{7.8}$$

We indicate this operator as $\mathcal{H}_\Gamma$. An alternative way to introduce $\mathcal{H}_\Gamma$ is via associated quadratic form

$$\mathfrak{q}_\Gamma(v_1, v_2) := (\nabla_{\widetilde{x}} v_1, \nabla_{\widetilde{x}} v_2)_{L_2(\Pi)} + (bv_1, v_2)_{L_2(\Gamma)}, \tag{7.9}$$

where $u \in W_{2,0}^1(\Pi)$ (see, for instance, [1, Appendix K, Sec. K.4.1], [4, Remark 4.1]). It is known that the operator $\mathcal{H}_\Gamma$ is self-adjoint.

We are going to show that there exists a diffeomorpism $x = \mathcal{P}(\widetilde{x})$ such that a unitary operator $\mathcal{U}$ defined by (7.5) maps $L_2(\Pi)$ onto $L_2(\Pi)$, and the operator $\mathcal{U}\mathcal{H}_\Gamma\mathcal{U}^{-1}$ has $W_{2,0}^2(\Pi)$ as the domain. We will also show that $\mathcal{U}\mathcal{H}_\Gamma\mathcal{U}^{-1} = -\Delta^{(D)} + \mathcal{L}$, where the operator $\mathcal{L}$ is given by (7.1).

First we introduce an auxiliary mapping as

$$\widetilde{\mathcal{P}}(\widetilde{x}) := \mathcal{P}_\Gamma^{-1}(\varrho, \xi), \quad \varrho := \widetilde{\varrho} + \frac{1}{2}\widetilde{\varrho}|\widetilde{\varrho}|b(\xi), \quad (\widetilde{\varrho}, \xi) = \mathcal{P}_\Gamma(x).$$

The coordinates $(\widetilde{\varrho}, \xi)$ are well-defined in a neighbourhood of $\Gamma$, which can be described as $\{x : |\widetilde{\varrho}| < \delta\}$, where $\delta$ is small enough. Indeed,

$$\widetilde{\mathcal{P}}_i(\widetilde{x}) = \widetilde{x}_i + \widetilde{\varrho}|\widetilde{\varrho}|\widehat{\mathcal{P}}_i(\widetilde{x}), \tag{7.10}$$

where the functions $\widehat{\mathcal{P}}_i(x)$ are continuously differentiable in a neighbourhood of $\Gamma$. Therefore, the Jacobian of $\widetilde{\mathcal{P}}_i$ tends to one as $\widetilde{\varrho} \to 0$ uniformly in $\xi \in \Gamma$. Now we construct the required mapping as follows

$$x = \mathcal{P}(\widetilde{x}) := \big(1 - \chi(\widetilde{\varrho}/\delta)\big)\widetilde{x} + \chi(\widetilde{\varrho}/\delta)\widetilde{\mathcal{P}}(\widetilde{x}). \tag{7.11}$$

The symbol $\widetilde{\chi}(t)$ indicates an even infinitely differentiable cut-off function being one as $|t|<1$ and vanishing as $|t|>2$. We assume also that $\delta$ is chosen so that $\operatorname{supp}\widetilde{\chi}(\widetilde{\varrho}/\delta)\cap\partial\Pi=\emptyset$. Let us prove that $\mathcal{P}$ is a $C^1$-diffeomorhism and $\mathcal{P}(\overline{\Pi})=\overline{\Pi}$.

It is obvious that $\mathcal{P}\in C^1(\overline{\Pi})$. If $\widetilde{\chi}(\widetilde{\varrho}/\delta)=0$, the mapping $\mathcal{P}$ acts as an identity mapping, and therefore for such $x$ the Jacobian $\mathrm{p}=\mathrm{p}(\widetilde{x})$ of $\mathcal{P}$ equals one. The function $\widetilde{\chi}$ is non-zero only in a small neighbourhood $\{\widetilde{x}:|\widetilde{\varrho}|<2\delta\}$ of $\Gamma$, where the identities (7.10) are applicable. These identities together with the definition of $\mathcal{P}$ imply that

$$\mathrm{p}(\widetilde{x})=1+\widetilde{\varrho}\mathrm{p}_1(\widetilde{x}),$$

where the function $\mathrm{p}_1(\widetilde{x})$ is bounded uniformly in $\delta$ and $\widetilde{x}$ as $|\widetilde{\varrho}|\leqslant 2\delta$. Hence, we can choose $\delta$ small enough so that $\mathrm{p}\geqslant 1/2$ as $|\widetilde{\varrho}|\leqslant 2\delta$. Therefore, $\mathcal{P}$ is $C^1$-diffeomorphism. As $x$ close to $\partial\Pi$, the diffeomorphism $\mathcal{P}$ acts as the identity mapping. It yields that $\mathcal{P}(\partial\Pi)=\partial\Pi$, $\mathcal{P}(\overline{\Pi})=\overline{\Pi}$. Since $\varrho=0$ as $\widetilde{\varrho}=0$, it follows that $\mathcal{P}(\Gamma)=\Gamma$.

The function $\widetilde{\varrho}\mapsto\varrho$ is continuously differentiable and its second derivative is piecewise continuous. Thus, the second derivatives of $\mathcal{P}_i(\widetilde{x})$ are piecewise continuous as well. The same is true for the inverse mapping $\mathcal{P}^{-1}$.

We introduce the unitary operator $\mathcal{U}$ by (7.5), where $\mathcal{P}$ is the diffeomorphism defined by (7.11). It is obvious that $\mathcal{U}$ maps $L_2(\Pi)$ onto itself. Let us prove that it maps the domain of $\mathcal{H}_\Gamma$ onto $W_{2,0}^2(\Pi)$. In order to do it, we have to study the behaviour of p in a vicinity of $\Gamma$. It is clear that $\mathrm{p}(\widetilde{x})\in C^2(\overline{\Pi}\setminus\Gamma)\cap C(\overline{\Pi})$, and this function has discontinuities at $\Gamma$ only. By $\mathrm{p}_\Gamma=\mathrm{p}_\Gamma(\widetilde{\varrho},\xi)$ we denote the Jacobian corresponding to $\mathcal{P}_\Gamma$. It is obvious that $\mathrm{p}_\Gamma\in C^2(\{(\widetilde{\varrho},\xi):|\widetilde{\varrho}|\leqslant\delta\})$. Employing the well-known properties of the Jacobians, we can express p in terms of $\mathrm{p}_\Gamma$,

$$\mathrm{p}(\widetilde{x})=\frac{(1+|\widetilde{\varrho}|b(\xi))\mathrm{p}_\Gamma(\widetilde{\varrho},\xi)}{\mathrm{p}_\Gamma(\varrho,\xi)}=\frac{\mathrm{p}_\Gamma(\widetilde{\varrho},\xi)(1+|\widetilde{\varrho}|b(\xi))}{\mathrm{p}_\Gamma(\widetilde{\varrho}+\frac{1}{2}\widetilde{\varrho}|\widetilde{\varrho}|b(\xi),\xi)},\quad |\widetilde{\varrho}|\leqslant\delta.$$

This relation allows us to conclude that $\mathrm{p}\in C^2(\{(\widetilde{\varrho},\xi):0\leqslant\widetilde{\varrho}\leqslant\delta\})$, $\mathrm{p}\in C^2(\{(\widetilde{\varrho},\xi):-\delta\geqslant\widetilde{\varrho}\geqslant 0\})$, and

$$\mathrm{p}^{1/2}\big|_{\widetilde{\varrho}=+0}-\mathrm{p}^{1/2}\big|_{\widetilde{\varrho}=-0}=0,\quad \frac{\partial}{\partial\widetilde{\varrho}}\mathrm{p}^{1/2}\big|_{\varrho=+0}-\frac{\partial}{\partial\widetilde{\varrho}}\mathrm{p}^{1/2}\big|_{\varrho=-0}=b. \tag{7.12}$$

Given $u=u(x)\in W_{2,0}^2(\Pi)$, we introduce the function $v=v(\widetilde{x}):=(\mathcal{U}^{-1}u)(\widetilde{x})=\mathrm{p}^{1/2}(\widetilde{x})u(\mathcal{P}(\widetilde{x}))$. Due to the smoothness of p, $v(\widetilde{x})\in W_2^2(\Pi\setminus\Gamma)\cap W_{2,0}^1(\Pi)$. The identities (7.12) and the belonging $u(x)\in W_2^2(\Pi)$ imply the condition (7.8) for $v$.

Suppose now that a function $v=v(\widetilde{x})\in W_2^2(\Pi\setminus\Gamma)\cap W_{2,0}^1(\Pi)$ satisfies (7.8). Due to the smoothness of $\mathcal{P}$ it is sufficient to check that the function $\mathcal{U}v$ regarded as depending on $\widetilde{x}$ belongs to $W_{2,0}^2(\Pi)$, i.e., $u(\widetilde{x}):=\mathrm{p}^{-1/2}(\widetilde{x})v(\widetilde{x})\in W_{2,0}^2(\Pi)$. It is clear that $u\in W_2^2(\Pi\setminus\Gamma)\cap W_{2,0}^1(\Pi)$. Hence, it remains to check the belonging $u\in W_2^2(\{\widetilde{x}:|\widetilde{\varrho}|\leqslant\delta\})$. The functions $\mathcal{P}_i$ being twice piecewise continuously differentiable, it is sufficient to make sure that the function $u$ treated as depending on $\widetilde{\varrho}$ and $\xi$ is an element of $W_2^2(\{(\widetilde{\varrho},\xi):|\widetilde{\varrho}|<\delta,\xi\in\Gamma\})$. We have $u\in W_2^1(R)$,

$u \in W_2^2(R_+)$, $u \in W_2^2(R_-)$, $R := \{(\widetilde{\varrho}, \xi) : |\widetilde{\varrho}| < \delta, \xi \in \Gamma\}$, $R_\pm := \{(\widetilde{\varrho}, \xi) : 0 < \pm\widetilde{\varrho} < \pm\delta, \xi \in \Gamma\}$. Hence, we have to prove the existence of the generalized second derivatives for $u$ belonging to $L_2(R)$. The condition (7.8) and the formulas (7.12) yield that

$$u\Big|_{\widetilde{\varrho}=+0} = u\Big|_{\widetilde{\varrho}=-0}, \quad \frac{\partial u}{\partial \widetilde{\varrho}}\Big|_{\widetilde{\varrho}=+0} = \frac{\partial u}{\partial \widetilde{\varrho}}\Big|_{\widetilde{\varrho}=-0}.$$

Since $u\Big|_{\widetilde{\varrho}=\pm 0} \in W_2^1(\Gamma)$, the first of these relations implies that

$$\frac{\partial u}{\partial \xi_i}\Big|_{\widetilde{\varrho}=+0} = \frac{\partial u}{\partial \xi_i}\Big|_{\widetilde{\varrho}=-0}, \quad i = 1, \dots, n-1.$$

Having the obtained relations in mind, for any $\zeta \in C_0^2(R)$ we integrate by parts,

$$\begin{aligned}\left(u, \frac{\partial^2 \zeta}{\partial \varrho^2}\right)_{L_2(R)} &= \left(u, \frac{\partial^2 \zeta}{\partial \varrho^2}\right)_{L_2(R_-)} + \left(u, \frac{\partial^2 \zeta}{\partial \varrho^2}\right)_{L_2(R_+)} = \\ &= \left(\frac{\partial^2 u}{\partial \varrho^2}, \zeta\right)_{L_2(R_-)} + \left(\frac{\partial^2 u}{\partial \varrho^2}, \zeta\right)_{L_2(R_+)},\end{aligned}$$

and in the same way we obtain:

$$\begin{aligned}\left(u, \frac{\partial^2 \zeta}{\partial \varrho \partial \xi_i}\right)_{L_2(R)} &= \left(\frac{\partial^2 u}{\partial \varrho \partial \xi_i}, \zeta\right)_{L_2(R_-)} + \left(\frac{\partial^2 u}{\partial \varrho \partial \xi_i}, \zeta\right)_{L_2(R_+)}, \\ \left(u, \frac{\partial^2 \zeta}{\partial \xi_i \partial \xi_j}\right)_{L_2(R)} &= \left(\frac{\partial^2 u}{\partial \xi_i \partial \xi_j}, \zeta\right)_{L_2(R_-)} + \left(\frac{\partial^2 u}{\partial \xi_i \partial \xi_j}, \zeta\right)_{L_2(R_+)}.\end{aligned}$$

Thus, the generalized second derivatives of the function $u$ exist and coincide with the corresponding derivatives of $u$ regarded as an element of $W_2^2(R_-)$ and $W_2^2(R_+)$.

Let us show that the operator $\mathcal{U}\mathcal{H}_\Gamma\mathcal{U}^{-1}$ can be represented as $\mathcal{U}\mathcal{H}_\Gamma\mathcal{U}^{-1} = -\Delta^{(D)} + \mathcal{L}$, where $\mathcal{L}$ is given by (7.1). Proceeding in the same way as in (7.7) and using (7.9), for any $u_1, u_2 \in C_0^\infty(\Pi)$ we obtain

$$(\mathcal{U}\mathcal{H}_\Gamma\mathcal{U}^{-1}u_1, u_2)_{L_2(\Pi)} = \int\limits_\Pi \left(\nabla_x \mathrm{p}^{1/2} u_1, \mathrm{p}^{-1}\mathrm{P}^t\mathrm{P}\nabla_x \mathrm{p}^{1/2} u_2\right)_{\mathbb{C}^n} \mathrm{d}x + \int\limits_\Gamma b u_1 \overline{u}_2 \,\mathrm{d}\xi.$$

We have used here that $\mathrm{p} \equiv 1$ as $x \in \Gamma$ and $\mathcal{P}(\Gamma) = \Gamma$. Employing (7.12) and having in mind that $\mathcal{P}\Big|_\Gamma = \mathrm{E}$ due to (7.10), we integrate by parts,

$$\begin{aligned}(\mathcal{U}\mathcal{H}_\Gamma\mathcal{U}^{-1}u_1, u_2)_{L_2(\Pi)} &= -\int\limits_\Gamma \overline{u}_2 \left(-\frac{\partial}{\partial \widetilde{\rho}}(\sqrt{\mathrm{p}}u_1)\Big|_{\widetilde{\rho}=+0} + \frac{\partial}{\partial \widetilde{\rho}}(\sqrt{\mathrm{p}}u_1)\Big|_{\widetilde{\rho}=-0} + bu_1\right) \mathrm{d}\xi - \\ &- \int\limits_{\Pi\setminus\Gamma} \overline{u}_2 \mathrm{p}^{1/2} \operatorname{div}_x \mathrm{p}^{-1}\mathrm{P}^t\mathrm{P}\nabla_x \mathrm{p}^{1/2} u_1 \,\mathrm{d}x = -\int\limits_{\Pi\setminus\Gamma} \overline{u}_2 \mathrm{p}^{1/2} \operatorname{div}_x \mathrm{p}^{-1}\mathrm{P}^t\mathrm{P}\nabla_x \mathrm{p}^{1/2} u_1 \,\mathrm{d}x.\end{aligned}$$

Here $\Pi\setminus\Gamma$ means that in a neighbourhood we partition the domain of integration into two pieces one being located in the set $\{x : |\widetilde{\varrho}| < 0\}$, while the other corresponds to $|\widetilde{\varrho}| > 0$. Such partition is needed since the first derivatives of p have jump at $\Gamma$ and therefore the second derivatives of p are not defined at $\Gamma$. At the same time, the function p has continuous second derivatives as $\widetilde{\varrho} < 0$ and these derivatives have finite limit as $\widetilde{\varrho} \to -0$. The same is true for $\widetilde{\varrho} > 0$.

The matrix P is piecewise continuously differentiable, and thus $\mathcal{U}\mathcal{H}_\Gamma\mathcal{U}^{-1} = -\mathrm{p}^{1/2}\operatorname{div}_x \mathrm{p}^{-1}\mathrm{P}^t\mathrm{P}\nabla_x\mathrm{p}^{1/2}$, where the second derivatives of p are treated in the aforementioned sense. This operator is obviously self-adjoint. Outside the set $\{x : |\widetilde{\varrho}| < 2\delta\}$ the diffeomorphism $\mathcal{P}$ acts as the identity mapping. It yields that at such points $\mathrm{P} = \mathrm{E}$, $\mathrm{p} = 1$. Therefore, $\mathcal{L}$ is a differential operator having compactly supported coefficients, and is a particular case of (7.1). It is also clear that the operator $\mathcal{L}$ satisfies (1.1). The inequality (1.2) can be checked in the same way how it was proved in the previous subsection.

**6. Integral operator.** The operator $\mathcal{L}$ is not necessary to be either a differential operator or reducible to a differential one. An example is an integral operator

$$\mathcal{L} = \int\limits_{\Omega} L(x, y)u(y)\,\mathrm{d}y.$$

The kernel $L \in L_2(\Pi\times\Pi)$ is assumed to be symmetric, i.e., $L(x, y) = \overline{L(y, x)}$. It is clear that the operator $\mathcal{L}$ satisfies (1.1), (1.2). It is also $\Delta^{(D)}$-compact and therefore the operators $\mathcal{H}$ and $\mathcal{H}_l$ are self-adjoint.

In conclusion we should stress that all possible examples of $\mathcal{L}$ are not exhausted by ones given above. For instance, combinations of these examples are possible like compactly supported magnetic field with delta interaction, delta interaction in a deformed waveguide, integro-differential operator, etc. Moreover, the operators $\mathcal{L}_-$ and $\mathcal{L}_+$ are not necessary to be of the same nature. For example, $\mathcal{L}_-$ can be a potential, while $\mathcal{L}_+$ describes compactly supported magnetic field with delta interaction.

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